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INDUCTIVE LIMITS IN THE OPERATOR SYSTEM AND RELATED CATEGORIES

LINDA MAWHINNEY AND IVAN G. TODOROV

ABSTRACT. We present a systematic development of inductive limits in the categories of ordered $*$ -vector spaces, Archimedean order unit spaces, matrix ordered spaces, operator systems and operator C^* -systems. We show that the inductive limit intertwines the operation of passing to the maximal operator system structure of an Archimedean order unit space, and that the same holds true for the minimal operator system structure if the connecting maps are complete order embeddings. We prove that the inductive limit commutes with the operation of taking the maximal tensor product with another operator system, and establish analogous results for injective functorial tensor products provided the connecting maps are complete order embeddings. We identify the inductive limit of quotient operator systems as a quotient of the inductive limit, in case the involved kernels satisfy a lifting condition, implied by complete biproximality. We describe the inductive limit of graph operator systems as operator systems of topological graphs, show that two such operator systems are completely order isomorphic if and only if their underlying graphs are isomorphic, identify the C^* -envelope of such an operator system, and prove a version of Glimm's Theorem on the isomorphism of UHF algebras in the category of operator systems.

CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. AOU spaces	4
2.2. Operator systems, operator spaces and tensor products	6
2.3. OMIN and OMAX	11
2.4. C^* -covers	11
2.5. Inductive limits	12
3. Inductive limits of AOU spaces	14
3.1. Inductive limits in the category OU	15
3.2. The state space of the inductive limit in OU	18
3.3. Inductive limits in the category AOU	19
4. Inductive limits of operator systems	23
4.1. Inductive limits of matrix ordered $*$ -vector spaces	23
4.2. Inductive limits of operator systems	26
4.3. Inductive limits of C^* -algebras	29

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4.4. Inductive limits of OMIN and OMAX	30
4.5. Inductive limits of universal C*-algebras	32
4.6. Quotients of inductive limits of operator systems	34
4.7. Inductive limits and tensor products	37
5. Inductive limits of operator C*-systems	47
6. Inductive limits of graph operator systems	50
6.1. Operator C*-systems in UHF algebras	51
6.2. Graphs associated to operator subsystems of UHF algebras	55
6.3. A graph isomorphism theorem	57
6.4. A generalisation of Glimm's theorem	59
References	61

1. INTRODUCTION

Operator systems were first studied in the late 1960s by Arveson [2]. Over the past five decades they have played a significant role in the development of non-commutative functional analysis and nowadays there is an extensive body of literature on their structure and properties [5, 8, 29].

Compared to the longer-studied category of C*-algebras, operator systems have the advantage to capture in a more subtle way properties of non-commutative order. It has become clear, for instance, that they can behave very differently than C*-algebras regarding the formation of categorical constructs such as tensor products [30, 32]. At the same time, their simpler structure allows one to express complexities of infinite-dimensional phenomena through finite-dimensional objects. For example, finite dimensional operator systems can be used to both reformulate the Connes Embedding Problem [19] and to characterise the weak expectation property of C*-algebras [15].

Inductive limits of C*-algebras first appeared over fifty years ago in [16], and have ever since occupied a prominent place in C*-algebra theory. In addition to their cornerstone role in Elliott's classification programme [13], they have been instrumental in applications to quantum physics, where questions of fundamental theoretical nature can be expressed in those terms [14]. In contrast, there is no similar development in the operator system category. While inductive limits of complete operator systems were introduced by Kirchberg in [21], and some very recent additions have been made through [24] and [23], no systematic study of operator system inductive limits and their applications has been conducted.

The purpose of this paper is to begin a systematic investigation of inductive limits of ordered *-vector spaces, operator systems and related categories, and to highlight some first applications. Our approach differs substantially from the one of [21]. Indeed, due to the emphasis on the development of approximation techniques, Kirchberg's interest lies in complete operator systems. Subsequently, his definition of the inductive limit relies on

the norm structure of the operator systems in question. Here, we are interested in the interactions between operator system structures and inductive limits, as well as in the tensor product theory, which was developed in [32] in the more general case of non-complete operator systems. This setting allows to infer all desired properties based on the (matrix) order through the Archimedeanisation techniques introduced in [33] and [32] and avoids to a substantial extent the use of norms.

The paper is organised as follows. After recalling some preliminary background in Section 2, we construct, in Section 3, the inductive limits in the categories of ordered $*$ -vector spaces and Archimedean order unit (AOU) spaces. We identify the state space of the inductive limit as the inverse limit of the corresponding state spaces. In Section 4, which is the core part of the paper, we develop in detail the inductive limit in the operator system category. We show that the OMAX operation, introduced in [32], is intertwined by the inductive limit construction. A similar result holds true for the OMIN operation, in the case the connecting morphisms are complete order embeddings. We identify the inductive limit of quotient operator systems as a quotient of an inductive limit, in case the involved kernels satisfy a lifting condition, implied by complete biproximality. We then establish a general intertwining theorem for the inductive limit and any injective functorial tensor products, provided the connecting maps are complete order embeddings. It applies to the minimal tensor product to give a result, recently established in the case of complete operator systems, in [24], and has as a consequence a corresponding commutation result for the commuting tensor product that was also highlighted, in the case of complete operator systems, in [24]. Although this general theorem does not apply to the maximal tensor product, we show that the inductive limit intertwines this tensor product as well. We also develop the inductive limit for the category of operator C^* -systems [29], that is, operator systems that are bimodules over a given C^* -algebra and whose matrix order structure is compatible with the module actions.

In Section 6, we consider inductive limits of graph operator systems. This class of operator systems was introduced in [11] and subsequently studied in [28], where the authors showed that the graph operator system is a complete isomorphism invariant for the corresponding graph, and identified its C^* -envelope. In view of the importance of graph operator systems in Quantum Information Theory, where they correspond to confusability graphs of quantum channels [11], we establish inductive limit versions of the aforementioned results. Namely, we show that the inductive limit of graph operator systems can be thought of as a topological graph operator system, and prove that two such operator systems are completely order isomorphic precisely when their underlying topological graphs are isomorphic. We also identify the C^* -envelope of such an operator system as the C^* -subalgebra of the surrounding UHF algebra, generated by the operator system. Finally, we prove an operator system version of the classical theorem of Glimm's that

characterises $*$ -isomorphism of two UHF algebras in terms of embeddings of the intermediate matrix algebras. Crucial for this section are Power's monograph [34] and the development of topological equivalence relations therein.

We close this section by pointing out that our results can be developed in the greater generality of inductive systems indexed by arbitrary nets, as opposed to sequences. In order to reduce the level of technicality and increase notational simplicity, we have decided to present the material in the setting of inductive sequences.

2. PRELIMINARIES

In this section, we gather necessary preliminary material that will be needed in the remainder of the paper.

2.1. AOU spaces. This subsection contains the basics on Archimedean order unit vector spaces, as developed by Paulsen and Tomforde in [33]. A $*$ -vector space is a complex vector space V equipped with an involution $*$, that is, a mapping $*$: $V \rightarrow V$ such that $x^{**} = x$, $(x + y)^* = x^* + y^*$ and $(\lambda x)^* = \bar{\lambda}x^*$ for all $x, y \in V$ and all $\lambda \in \mathbb{C}$. Let $V_h = \{x \in V : x^* = x\}$ and call the elements of V_h *hermitian*. For any $x \in V$, we have that

$$x = \operatorname{Re}(x) + i\operatorname{Im}(x),$$

where

$$\operatorname{Re}(x) = \frac{x + x^*}{2} \text{ and } \operatorname{Im}(x) = \frac{x - x^*}{2i}$$

are hermitian. An *ordered $*$ -vector space* is a pair (V, V^+) , where V is a $*$ -vector space and V^+ is a cone in V_h (that is, a subset of V_h closed under addition and multiplication by a non-negative scalar) such that $V^+ \cap (-V^+) = \{0\}$. For $x, y \in V$, we write $x \leq y$ if $y - x \in V^+$.

Let (V, V^+) be an ordered $*$ -vector space. We say that $e \in V^+$ is an *order unit* for V if for every $x \in V_h$, there exists $r > 0$ such that $x \leq re$. We call the triple (V, V^+, e) an *order unit space*. An order unit $e \in V^+$ is called *Archimedean* if

$$V^+ = \{x \in V_h : re + x \in V^+ \text{ for all } r > 0\}.$$

A triple (V, V^+, e) , where (V, V^+) is an ordered $*$ -vector space for which e is an Archimedean order unit, is called an *Archimedean order unit space* (or an *AOU space*, for short). We will often denote by e_V the order unit with which an ordered $*$ -vector space (V, V^+) is equipped.

Let (V, V^+) and (W, W^+) be order unit spaces. A linear map $\phi : V \rightarrow W$ is called *positive* if $\phi(V^+) \subseteq W^+$ and *unital* if $\phi(e_V) = e_W$. The map $\phi : V \rightarrow W$ is called an *order isomorphism* if it is bijective and $v \in V^+$ if and only if $\phi(v) \in W^+$. The complex field \mathbb{C} will henceforth be equipped with its standard AOU space structure, and linear maps $f : V \rightarrow \mathbb{C}$ will as usual be referred to as *functionals*. A *state* on V is a unital positive functional.

We write $S(V)$ for the set of all states on V and call it the *state space*; note that $S(V)$ is a cone.

Let (V, V^+) be an ordered $*$ -vector space with order unit e . We introduce a seminorm on V_h , letting

$$\|x\|_h = \inf\{r \in \mathbb{R} : -re \leq x \leq re\}, \quad x \in V_h.$$

We call $\|\cdot\|_h$ the *order seminorm* on V_h determined by e . We note that $\|\cdot\|_h$ is a norm if e is Archimedean [33, Proposition 2.23]. An *order seminorm* $\|\cdot\|$ on V is a seminorm such that $\|x^*\| = \|x\|$ for all $x \in V$ and $\|x\| = \|x\|_h$ for all $x \in V_h$. The set of order seminorms on V has a maximal and a minimal (with respect to point-wise ordering) element. The minimal seminorm is given by letting

$$(1) \quad \|x\| = \sup\{|f(x)| : f \in S(V)\},$$

and all order seminorms are equivalent to it. If (V, V^+, e) is an AOU space, $\|\cdot\|$ is in fact a norm.

By (1), the states of V are contractive with respect to the minimal order seminorm. We denote by V' the space of all functionals continuous in the topology defined by any of the order seminorms. If V_1 and V_2 are ordered $*$ -vector spaces with order units and $\phi : V_1 \rightarrow V_2$ is a unital positive map then we let $\phi' : V_2' \rightarrow V_1'$ be the dual of ϕ . If (V, V^+, e) is an AOU space then $S(V)$ is a compact topological space with respect to the weak* topology inherited from the topology generated by any order norm on V . Thus, the map $\phi : V \rightarrow C(S(V))$, given by $\langle \phi(x), f \rangle = \langle f, x \rangle$, is a unital injective map that is an order isomorphism onto its image. Furthermore, ϕ is an isometry with respect to the minimal order norm on V and the uniform norm on $C(S(V))$. The latter statement can be viewed as a complex version of Kadison's representation theorem (see [18] or [1, Theorem II.1.8]); for a proof, we refer the reader to [33, Theorem 5.2].

The proof of the next remark is straightforward and is omitted.

Remark 2.1. Let V be an ordered $*$ -vector space with order unit and let $\|\cdot\|'$ be an order seminorm on V . If $x \in V$, we have that $\|x\|' = 0$ if and only if $\|\operatorname{Re}(x)\|_h = 0$ and $\|\operatorname{Im}(x)\|_h = 0$.

In order to avoid excessive notation, we will sometimes denote the ordered $*$ -vector space (V, V^+, e) simply by V .

Let us denote by **OU** the category whose objects are ordered $*$ -vector spaces with order units and whose morphisms are unital positive maps, and by **AOU** the category whose objects are AOU spaces and whose morphisms are unital positive maps. Clearly, we have a forgetful functor $\mathbf{F} : \mathbf{AOU} \rightarrow \mathbf{OU}$. In [33, Section 3.2], the process of Archimedeanisation is discussed which provides us with a left adjoint to this functor. Let (V, V^+, e) be an ordered $*$ -vector space with order unit. Define

$$D = \{v \in V_h : re + v \in V^+ \text{ for every } r > 0\}$$

and

$$(2) \quad N = \{v \in V : f(v) = 0 \text{ for all } f \in S(V)\}.$$

Clearly, D is a cone, while N is a linear subspace of V . Equip V/N with the involution given by $(v + N)^* = v^* + N$, and set

$$(V/N)^+ = \{v + N : v \in D\}.$$

It was shown in [33, Theorem 3.16] that $(V/N, (V/N)^+, e + N)$ is an AOU space, which was called therein the *Archimedeanisation* of (V, V^+, e) and denoted by V_{Arch} . It satisfies the following universal property.

Theorem 2.2. *Let V be an ordered $*$ -vector space with order unit. The quotient map $\varphi : V \rightarrow V_{\text{Arch}}$ is the unique unital surjective positive linear map from V onto V_{Arch} such that, whenever W is an Archimedean order unit space and $\phi : V \rightarrow W$ is a unital positive linear map, then there exists a unique positive linear map $\phi_{\text{Arch}} : V_{\text{Arch}} \rightarrow W$ such that $\phi = \phi_{\text{Arch}} \circ \varphi$.*

Furthermore, if $(\tilde{V}, \tilde{\varphi})$ is a pair consisting of an AOU space \tilde{V} and a unital surjective positive linear map $\tilde{\varphi} : V \rightarrow \tilde{V}$ such that, whenever W is an Archimedean order unit space and $\phi : V \rightarrow W$ is a unital positive linear map there exists a unique positive linear map $\tilde{\phi} : \tilde{V} \rightarrow W$ with $\phi = \tilde{\phi} \circ \tilde{\varphi}$, then there exists a unital order isomorphism $\psi : V_{\text{Arch}} \rightarrow \tilde{V}$ such that $\psi \circ \varphi = \tilde{\varphi}$.

2.2. Operator systems, operator spaces and tensor products.

2.2.1. Basic concepts. For a vector space \mathcal{S} , we let $M_{n,m}(\mathcal{S})$ be the vector space of all n by m matrices with entries in \mathcal{S} . We set $M_{n,m} = M_{n,m}(\mathbb{C})$, $M_n(\mathcal{S}) = M_{n,n}(\mathcal{S})$ and $M_n = M_{n,n}$.

Let \mathcal{S} be a $*$ -vector space. We equip $M_n(\mathcal{S})$ with the involution $(s_{i,j})_{i,j}^* = (s_{j,i}^*)_{i,j}$; it turns $M_n(\mathcal{S})$ into a $*$ -vector space. A family $\{C_n\}_{n \in \mathbb{N}}$, where $C_n \subseteq M_n(V)$, is called a *matrix ordering* on \mathcal{S} if

- (i) C_n is a cone in $M_n(\mathcal{S})_h$ for each $n \in \mathbb{N}$,
- (ii) $C_n \cap (-C_n) = \{0\}$ for each $n \in \mathbb{N}$, and
- (iii) for each $n, m \in \mathbb{N}$ and each $\alpha \in M_{n,m}$ we have that $\alpha^* C_n \alpha \subseteq C_m$.

A *matrix ordered* $*$ -vector space is a pair $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}})$ where \mathcal{S} is a $*$ -vector space and $\{C_n\}_{n \in \mathbb{N}}$ is a matrix ordering. We refer to condition (iii) as the *compatibility* of the family $\{C_n\}_{n \in \mathbb{N}}$ and often write $M_n(\mathcal{S})^+$ for C_n .

Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}})$ be a matrix ordered $*$ -vector space. For each $e \in \mathcal{S}$ and $n \in \mathbb{N}$, let

$$e^{(n)} := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix} \in M_n(\mathcal{S}),$$

where the off-diagonal entries are zero. We say that $e \in C_1$ is a *matrix order unit* if $e^{(n)}$ is an order unit for $(M_n(\mathcal{S}), C_n)$ for all $n \in \mathbb{N}$. Similarly, we say that e is an *Archimedean matrix order unit* if $e^{(n)}$ is an Archimedean order unit for $(M_n(\mathcal{S}), C_n)$ for each $n \in \mathbb{N}$. An *operator system* is a matrix ordered $*$ -vector space with an Archimedean matrix order unit. In order to

avoid excessive notation, we will sometimes denote the triple $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e)$ simply by \mathcal{S} ; the unit is denoted by $e_{\mathcal{S}}$ if there is risk of confusion.

Let \mathcal{S} and \mathcal{T} be matrix ordered $*$ -vector spaces with matrix order units and $\phi : \mathcal{S} \rightarrow \mathcal{T}$ be a linear map. We define $\phi^{(n)} : M_n(\mathcal{S}) \rightarrow M_n(\mathcal{T})$ by letting

$$\phi^{(n)} \left(\begin{pmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & & \vdots \\ s_{n,1} & \cdots & s_{n,n} \end{pmatrix} \right) = \begin{pmatrix} \phi(s_{1,1}) & \cdots & \phi(s_{1,n}) \\ \vdots & & \vdots \\ \phi(s_{n,1}) & \cdots & \phi(s_{n,n}) \end{pmatrix},$$

$n \in \mathbb{N}$. We say that ϕ is n -positive if $\phi^{(n)}$ is positive and that ϕ is *completely positive* if ϕ is n -positive for all $n \in \mathbb{N}$. Furthermore, we say that ϕ is a *complete order isomorphism* if ϕ is a bijection and both ϕ and ϕ^{-1} are completely positive. We say that ϕ is a *unital complete order embedding* if ϕ is a unital complete order isomorphism onto its image.

Let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators on a Hilbert space \mathcal{H} . If \mathcal{S} is a subset of $\mathcal{B}(\mathcal{H})$, we set $\mathcal{S}^* = \{s \in \mathcal{S} : s^* \in \mathcal{S}\}$ and call \mathcal{S} *selfadjoint* if $\mathcal{S} = \mathcal{S}^*$. A *concrete operator system* is a selfadjoint subspace of $\mathcal{B}(\mathcal{H})$ which contains the identity operator I . If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a concrete operator system then $M_n(\mathcal{S}) \subseteq M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(n)})$ and therefore $M_n(\mathcal{S})$ inherits an order structure from $\mathcal{B}(\mathcal{H}^{(n)})$. Note that I is an Archimedean matrix order unit for the matrix order structure thus defined. Hence, a concrete operator system is an operator system. The following fundamental result [8, Theorem 4.4] establishes the converse.

Theorem 2.3 (Choi–Effros [8]). *Let $(V, \{C_n\}_{n \in \mathbb{N}}, e)$ be an operator system. Then there exists a Hilbert space \mathcal{H} , a concrete operator system $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ and a complete order isomorphism $\Phi : V \rightarrow \mathcal{S}$ such that $\Phi(e) = I$.*

2.2.2. Operator spaces. Let \mathcal{X} be a Banach space and $\|\cdot\|_n$ be a norm on $M_n(\mathcal{X})$, $n \in \mathbb{N}$. We call $(\mathcal{X}, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$ an *operator space* if the following are satisfied:

- (i) $\left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}$ and
- (ii) $\|\alpha X \beta\|_n \leq \|\alpha\| \|X\|_n \|\beta\|$

for all $X \in M_n(\mathcal{X})$, $Y \in M_m(\mathcal{X})$ and $\alpha, \beta \in M_n$.

Let $(\mathcal{X}, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$ and $(\mathcal{Y}, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$ be operator spaces and let $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ be a linear map. We let $\|\phi\|_{\text{cb}} = \sup\{\|\phi^{(n)}\| : n \in \mathbb{N}\}$ and say that ϕ is *completely bounded* if $\|\phi\|_{\text{cb}}$ is finite, ϕ is *completely contractive* if $\|\phi\|_{\text{cb}} \leq 1$, and a *complete isometry* if $\phi^{(n)}$ is an isometry for every n .

Let us denote by **OSp** the category whose objects are operator spaces and whose morphisms are completely bounded maps. If \mathcal{X} is an operator space and \mathcal{X}' is the Banach space dual of \mathcal{X} then there is a natural way to induce an operator space structure on \mathcal{X}' as follows [4] (for more details see [12, Section 3.2]). If $S = (s_{i,j})_{i,j} \in M_n(\mathcal{X}')$ then S determines a linear mapping $\tilde{S} : \mathcal{X} \rightarrow M_n$, given by $\tilde{S}(x) = (\langle s_{i,j}, x \rangle)_{i,j}$; we set $\|S\|_n = \|\tilde{S}\|_{\text{cb}}$.

It follows from the Choi-Effros Theorem that every operator system is, canonically, an operator space. The following result [29, Lemma 3.1] provides a characterisation of the norm in operator systems in terms of the matrix order structure.

Lemma 2.4. *Let \mathcal{S} be an operator system and $x \in M_n(\mathcal{S})$. Then $\|x\| \leq 1$ if and only if $\begin{pmatrix} 1_n & x \\ x^* & 1_n \end{pmatrix} \in M_{2n}(\mathcal{S})^+$.*

If ϕ is a unital map between operator systems then ϕ is completely contractive if and only if ϕ is completely positive (see [29, Proposition 3.6]). Thus, a unital linear map between operator systems is a unital complete isometry if and only if it is a unital complete order embedding.

It is proved in [33] that if \mathcal{A} is a unital C*-algebra, then its C*-norm is an order norm. Therefore, if \mathcal{S} is an operator system, the operator system norm on \mathcal{S} is an order norm. Indeed, if we choose a unital C*-algebra \mathcal{A} with unit $e_{\mathcal{A}}$ such that $\phi : \mathcal{S} \rightarrow \mathcal{A}$ is a unital complete order embedding (see Theorem 2.3 for the existence of \mathcal{A} and ϕ), then for any $r \in \mathbb{R}$ and $s \in \mathcal{S}$,

$$-re_{\mathcal{S}} \leq s \leq re_{\mathcal{S}} \text{ if and only if } -re_{\mathcal{A}} = \phi(-re_{\mathcal{S}}) \leq \phi(s) \leq \phi(re_{\mathcal{S}}) = re_{\mathcal{A}}.$$

Therefore for any $s \in \mathcal{S}_h$,

$$\begin{aligned} \|s\|_{\mathcal{S}} = \|\phi(s)\|_{\mathcal{A}} &= \inf\{r \in \mathbb{R} : -re_{\mathcal{A}} \leq \phi(s) \leq re_{\mathcal{A}}\} \\ &= \inf\{r \in \mathbb{R} : -re_{\mathcal{S}} \leq s \leq re_{\mathcal{S}}\}. \end{aligned}$$

2.2.3. Operator system tensor products. Suppose that $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e_{\mathcal{S}})$ and $(\mathcal{T}, \{D_n\}_{n \in \mathbb{N}}, e_{\mathcal{T}})$ are operator systems. We denote by $\mathcal{S} \odot \mathcal{T}$ their algebraic tensor product. We call a family $\mu = \{P_n\}_{n \in \mathbb{N}}$ of cones, where $P_n \subseteq M_n(\mathcal{S} \odot \mathcal{T})$, an *operator system structure* on $\mathcal{S} \odot \mathcal{T}$ if it satisfies the following properties:

- (i) $(\mathcal{S} \odot \mathcal{T}, \{P_n\}_{n \in \mathbb{N}}, e_{\mathcal{S}} \otimes e_{\mathcal{T}})$ is an operator system, denoted $\mathcal{S} \otimes_{\mu} \mathcal{T}$,
- (ii) $C_n \otimes D_m \subseteq P_{nm}$ for all $n, m \in \mathbb{N}$, and
- (iii) if $m, n \in \mathbb{N}$ and $\phi : \mathcal{S} \rightarrow M_n$, $\psi : \mathcal{T} \rightarrow M_m$ are unital completely positive maps then $\phi \otimes \psi : \mathcal{S} \otimes_{\mu} \mathcal{T} \rightarrow M_{nm}$ is a completely positive map.

An *operator system tensor product* [20] is a mapping $\mu : \mathbf{OS} \times \mathbf{OS} \rightarrow \mathbf{OS}$ such that $\mu(\mathcal{S}, \mathcal{T})$ is an operator system with an underlying space $\mathcal{S} \odot \mathcal{T}$ for every $\mathcal{S}, \mathcal{T} \in \mathbf{OS}$. Associativity is not assumed as a part of this definition, and although the main instances of tensor products we will use are associative, this will bear no special significance in our results. We call an operator system tensor product *functorial* if for any four operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and \mathcal{T}_2 we have that if $\phi_1 : \mathcal{S}_1 \rightarrow \mathcal{T}_1$ and $\phi_2 : \mathcal{S}_2 \rightarrow \mathcal{T}_2$ are unital completely positive maps then $\phi_1 \otimes \phi_2 : \mathcal{S}_1 \otimes_{\mu} \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes_{\mu} \mathcal{T}_2$ is a unital completely positive map. An operator system tensor product is *injective* if whenever ϕ_1 and ϕ_2 are unital complete order embeddings then $\phi_1 \otimes \phi_2$ is a unital complete order embedding. Let \mathcal{T} be an operator system and μ be an operator system tensor product. We say that \mathcal{T} is μ -*injective* if for any

pair of operator systems \mathcal{S}_1 and \mathcal{S}_2 we have that if $\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is a unital complete order embedding then $\phi \otimes \text{id}_{\mathcal{T}} : \mathcal{S}_1 \otimes_{\mu} \mathcal{T} \rightarrow \mathcal{S}_2 \otimes_{\mu} \mathcal{T}$ is a unital complete order embedding.

Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e_{\mathcal{S}})$ and $(\mathcal{T}, \{D_n\}_{n \in \mathbb{N}}, e_{\mathcal{T}})$ be operator systems and let $\iota_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$ and $\iota_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{B}(\mathcal{K})$ be unital complete order embeddings. The *minimal operator system tensor product* $\mathcal{S} \otimes_{\min} \mathcal{T}$ of \mathcal{S} and \mathcal{T} has operator system structure arising from the embedding $\iota_{\mathcal{S}} \otimes \iota_{\mathcal{T}} : \mathcal{S} \odot \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. It is proved in [20, Theorem 4.4] that the minimal operator system tensor product is injective, functorial, and independent of the concrete embeddings of \mathcal{S} and \mathcal{T} .

Let

$$P_n^{\max}(\mathcal{S}, \mathcal{T}) := \{\alpha(C \otimes D)\alpha^* : C \in C_k, D \in D_m, \alpha \in M_{n,km}, k, m \in \mathbb{N}\}.$$

The *maximal operator system tensor product* of \mathcal{S} and \mathcal{T} , denoted $\mathcal{S} \otimes_{\max} \mathcal{T}$, is the Archimedeanisation of $(\mathcal{S} \otimes \mathcal{T}, \{P_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n \in \mathbb{N}}, e_{\mathcal{S}} \otimes e_{\mathcal{T}})$ (we refer the reader to Subsection 2.2.4 below for the description of this construction).

Let \mathcal{H} be a Hilbert space. A bilinear map $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is called *jointly completely positive* if, for all $P = (x_{i,j}) \in C_n$ and $Q = (y_{k,l}) \in D_m$, the matrix $\phi^{(n,m)}(P, Q) := (\phi(x_{i,j}, y_{k,l}))$ is a positive element of $M_{nm}(\mathcal{B}(\mathcal{H}))$.

Theorem 2.5. *Let \mathcal{S} and \mathcal{T} be operator systems. If $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is a jointly completely positive map, then its linearisation $\phi_L : \mathcal{S} \otimes_{\max} \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is completely positive.*

Furthermore if μ is an operator system structure on $\mathcal{S} \odot \mathcal{T}$ with the property that the linearisation of every jointly completely positive map $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is completely positive on $\mathcal{S} \otimes_{\mu} \mathcal{T}$ then $\mathcal{S} \otimes_{\mu} \mathcal{T} = \mathcal{S} \otimes_{\max} \mathcal{T}$.

The *commuting operator system tensor product* is the operator system arising from the inclusion of $\mathcal{S} \odot \mathcal{T}$ into $C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$, denoted $\mathcal{S} \otimes_c \mathcal{T}$ (see Subsection 2.4 for the definition of the universal C*-algebra $C_u^*(\mathcal{R})$ of an operator system \mathcal{R}). It is proved in [20, Theorem 5.5 and Theorem 6.3] that the maximal operator system tensor product and the commuting operator system tensor product are both functorial.

2.2.4. The Archimedeanisation of matrix ordered *-vector spaces. The process of Archimedeanisation for matrix ordered *-vector spaces was described in [32, Section 3.1]. Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e)$ be a matrix ordered *-vector space with matrix order unit. For each $n \in \mathbb{N}$, set

$$N_n = \{S \in M_n(\mathcal{S}) : f(S) = 0 \text{ for all } f \in S(M_n(\mathcal{S}))\}.$$

Recall the notation from (2); it is proved in [32, Lemma 3.14] that $M_n(N) = N_n$, $n \in \mathbb{N}$. Define

$$C_n^{\text{Arch}} = \left\{ S + M_n(N) \in (M_n(\mathcal{S})/M_n(N))_h : \right. \\ \left. re^{(n)} + S + M_n(N) \in C_n + M_n(N) \text{ for all } r > 0 \right\}.$$

Then $(\mathcal{S}/N, \{C_n^{\text{Arch}}\}_{n \in \mathbb{N}}, e + N)$ is an operator system. We call this the *Archimedeanisation* of \mathcal{S} and denote it by $\mathcal{S}_{\text{Arch}}$. It has the following universal property.

Theorem 2.6 ([32]). *Let \mathcal{S} be a matrix ordered $*$ -vector space with matrix order unit. The quotient map $\varphi : \mathcal{S} \rightarrow \mathcal{S}_{\text{Arch}}$ is the unique unital surjective completely positive map such that, whenever \mathcal{T} is an operator system and $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map, there exists a unique completely positive map $\phi_{\text{Arch}} : \mathcal{S}_{\text{Arch}} \rightarrow \mathcal{T}$ such that $\phi = \phi_{\text{Arch}} \circ \varphi$.*

Furthermore, if $(\tilde{\mathcal{S}}, \tilde{\varphi})$ is a pair consisting of an operator system and unital surjective completely positive map $\tilde{\varphi} : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$ with the property that, whenever \mathcal{T} is an operator system and $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map, there exists a unique completely positive map $\tilde{\phi} : \tilde{\mathcal{S}} \rightarrow \mathcal{T}$ such that $\phi = \tilde{\phi} \circ \tilde{\varphi}$, then there exists a unital complete order isomorphism $\psi : \mathcal{S}_{\text{Arch}} \rightarrow \tilde{\mathcal{S}}$ such that $\psi \circ \varphi = \tilde{\varphi}$.

Remark 2.7. It is shown in [32, Remark 3.17] that the Archimedeanisation of a matrix ordered $*$ -vector space with matrix order unit is precisely the operator system formed by taking the Archimedeanisation at every matrix level.

We will make use of the following facts in the sequel.

Lemma 2.8. *Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e)$ be an operator system, V be a vector space and $\phi : \mathcal{S} \rightarrow V$ be an injective linear map. Equip $\phi(\mathcal{S})$ with the involution given by $\phi(x)^* \stackrel{\text{def}}{=} \phi(x^*)$. Then $(\phi(\mathcal{S}), \{\phi^{(n)}(C_n)\}_{n \in \mathbb{N}}, \phi(e))$ is an operator system.*

Proof. The facts that the family $\{\phi^{(n)}(C_n)\}_{n \in \mathbb{N}}$ is compatible and that $\phi(e)$ is matrix order unit are straightforward. To show that $\phi^{(n)}(e^{(n)})$ is Archimedean, suppose that $x \in M_n(\mathcal{S})$ is a selfadjoint element such that $\phi^{(n)}(x) + r\phi^{(n)}(e^{(n)}) \in \phi(C_n)$ for all $r > 0$. By the injectivity of ϕ , $x + re^{(n)} \in C_n$ for all $r > 0$, and hence $x \in C_n$. Thus, $\phi^{(n)}(x) \in \phi^{(n)}(C_n)$ and the proof is complete. \square

Lemma 2.9. *Let \mathcal{S}, \mathcal{T} and \mathcal{P} be operator systems and let $\phi : \mathcal{S} \rightarrow \mathcal{T}$ and $\psi : \mathcal{T} \rightarrow \mathcal{P}$ be unital linear maps. If ψ and $\psi \circ \phi$ are complete order embeddings then ϕ is a complete order embedding.*

Proof. Let $n \in \mathbb{N}$ and $S \in M_n(\mathcal{S})^+$. Then $\psi^{(n)} \circ \phi^{(n)}(S) \in M_n(\mathcal{P})^+$ and therefore $\phi^{(n)}(S) \in M_n(\mathcal{T})^+$. Suppose that $S \in M_n(\mathcal{S})$ and $\phi^{(n)}(S) \in M_n(\mathcal{T})^+$. Then $\psi^{(n)} \circ \phi^{(n)}(S) \in M_n(\mathcal{P})^+$ and therefore $S \in M_n(\mathcal{S})^+$. \square

We denote by **MOU** (resp. **OS**) the category whose objects are matrix ordered $*$ -vector spaces with matrix order unit (resp. operator systems) and whose morphisms are unital completely positive maps.

2.3. OMIN and OMAX. Let (V, V^+, e) be an AOU space. An *operator system structure* on (V, V^+, e) is a family $\{P_n\}_{n \in \mathbb{N}}$ such that $(V, \{P_n\}_{n \in \mathbb{N}}, e)$ is an operator system and $P_1 = V^+$. There are two extremal operator system structures [32] that will play a significant role in the sequel. The *minimal* operator system structure on (V, V^+, e) is the family $\{C_n^{\min}(V)\}_{n \in \mathbb{N}}$, where

$$C_n^{\min}(V) = \left\{ (x_{i,j})_{i,j} \in M_n(V) : \sum_{i,j=1}^n \bar{\lambda}_i \lambda_j x_{i,j} \in V^+ \text{ for all } \lambda_1, \dots, \lambda_n \in \mathbb{C} \right\}.$$

We set $\text{OMIN}(V) = (V, \{C_n^{\min}(V)\}_{n \in \mathbb{N}}, e)$.

Theorem 2.10 ([32]). *Let (V, V^+, e) be an AOU space and $n \in \mathbb{N}$. Then $(x_{i,j})_{i,j} \in C_n^{\min}(V)$ if and only if $(\langle f, x_{i,j} \rangle)_{i,j} \in M_n^+$ for each $f \in S(V)$.*

It follows from Theorem 2.10 that $\text{OMIN}(V)$ is the operator system induced by the canonical inclusion of V into $C(S(V))$.

We define $\text{OMAX}(V)$ to be the Archimedeanisation of the matrix ordered space $(V, \{D_n^{\max}(V)\}_{n \in \mathbb{N}}, e)$, where

$$D_n^{\max}(V) = \left\{ \sum_{i=1}^k a_i \otimes x_i : x_i \in V^+, a_i \in M_n^+, i = 1, \dots, k, k \in \mathbb{N} \right\}.$$

Let $\mathbf{F} : \mathbf{OS} \rightarrow \mathbf{AOU}$ be the forgetful functor. It can be seen that $\text{OMIN} : \mathbf{AOU} \rightarrow \mathbf{OS}$ is a right adjoint functor to \mathbf{F} and $\text{OMAX} : \mathbf{AOU} \rightarrow \mathbf{OS}$ is a left adjoint functor to \mathbf{F} (see [25] for relevant background in Category Theory).

2.4. C*-covers. Let \mathcal{S} be an operator system. A *C*-cover* is a pair (\mathcal{A}, ν) consisting of a unital C*-algebra and a unital completely isometric embedding $\nu : \mathcal{S} \rightarrow \mathcal{A}$ such that $\nu(\mathcal{S})$ generates \mathcal{A} as a C*-algebra. The *universal C*-cover* $(C_u^*(\mathcal{S}), \iota)$ of \mathcal{S} was defined in [22] and is characterised by the following universal property:

Proposition 2.11. *Let \mathcal{S} be an operator system, \mathcal{A} be a C*-algebra and $\phi : \mathcal{S} \rightarrow \mathcal{A}$ be a unital completely positive map. Then there exists a *-homomorphism*

$$\tilde{\phi} : C_u^*(\mathcal{S}) \rightarrow \mathcal{A}$$

such that $\tilde{\phi} \circ \iota = \phi$. Moreover, if (\mathcal{B}, μ) is another C-cover of \mathcal{S} such that, whenever \mathcal{A} is a C*-algebra and $\phi : \mathcal{S} \rightarrow \mathcal{A}$ is a unital completely positive map, there exists a *-homomorphism*

$$\tilde{\phi} : \mathcal{B} \rightarrow \mathcal{A}$$

*such that $\tilde{\phi} \circ \mu = \phi$, then there exists a *-isomorphism $\rho : \mathcal{B} \rightarrow C_u^*(\mathcal{S})$ with $\rho \circ \mu = \iota$.*

We call $C_u^*(\mathcal{S})$ the *universal C*-algebra of \mathcal{S}* . The *C*-envelope* of \mathcal{S} , introduced in [17] (see also [5, Section 4.3]) is, on the other hand, the C*-cover $(C_e^*(\mathcal{S}), \kappa)$, characterised by the following universal property: if (\mathcal{A}, ϕ)

is a C^* -cover of \mathcal{S} , then there exists a $*$ -homomorphism

$$\tilde{\phi} : \mathcal{A} \rightarrow C_e^*(\mathcal{S})$$

such that $\tilde{\phi} \circ \phi = \kappa$. Clearly, the pair $(C_e^*(\mathcal{S}), \kappa)$ is unique in the sense that if (\mathcal{B}, μ) is another pair with the same property then there exists a $*$ -isomorphism $\rho : \mathcal{B} \rightarrow C_e^*(\mathcal{S})$ with $\rho \circ \mu = \kappa$. We note that the C^* -algebras $C_u^*(\mathcal{S})$ and $C_e^*(\mathcal{S})$ are rarely $*$ -isomorphic (in particular, this never happens when \mathcal{S} is a C^* -algebra itself [22]), and we refer the reader to [22] for more information on these C^* -covers.

The following fact is a straightforward consequence of the universal property of C^* -envelopes.

Remark 2.12. Let \mathcal{S} and \mathcal{T} be operator systems and let $(C_e^*(\mathcal{S}), \iota_{\mathcal{S}})$ and $(C_e^*(\mathcal{T}), \iota_{\mathcal{T}})$ be the C^* -envelopes of \mathcal{S} and \mathcal{T} , respectively. If $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital complete order isomorphism, then there exists a unital $*$ -isomorphism $\rho : C_e^*(\mathcal{S}) \rightarrow C_e^*(\mathcal{T})$ such that $\rho \circ \iota_{\mathcal{S}} = \iota_{\mathcal{T}} \circ \phi$.

2.5. Inductive limits. We recall some basic categorical notions which will be necessary in the sequel; we refer the reader to [25] for further details.

Definition 2.13. Let \mathbf{C} be a category. An inductive system in \mathbf{C} is a pair $(\{A_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}})$ where A_k is an object in \mathbf{C} and $\alpha_k : A_k \rightarrow A_{k+1}$ is a morphism for each $k \in \mathbb{N}$. An inductive limit of the inductive system $(\{A_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}})$ is a pair $(A, \{\alpha_{k,\infty}\}_{k \in \mathbb{N}})$ where A is an object in \mathbf{C} and $\alpha_{k,\infty} : A_k \rightarrow A$ is a morphism, $k \in \mathbb{N}$, such that

- (i) $\alpha_{k+1,\infty} \circ \alpha_k = \alpha_{k,\infty}$, $k \in \mathbb{N}$, and
- (ii) if $(B, \{\beta_k\}_{k \in \mathbb{N}})$ is another pair such that B is an object in \mathbf{C} , $\beta_k : A_k \rightarrow B$ is a morphism and $\beta_{k+1} \circ \alpha_k = \beta_k$, $k \in \mathbb{N}$, then there exists a unique morphism $\mu : A \rightarrow B$ such that $\mu \circ \alpha_{k,\infty} = \beta_k$, $k \in \mathbb{N}$.

Suppose that $(\{A_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}})$ is an inductive system. If it exists, its inductive limit is unique and will be denoted by $\varinjlim_{\mathbf{C}} (A_k, \alpha_k)$ or $\varinjlim_{\mathbf{C}} A_k$ when the context is clear. We will refer to α_k , $k \in \mathbb{N}$, as the *connecting morphisms*, and set

$$\alpha_{k,l} = \alpha_{l-1} \circ \cdots \circ \alpha_k \text{ if } k < l \text{ and } \alpha_{k,k} = \text{id}_{A_k};$$

we thus have that $\alpha_{k,l}$ is a morphism from A_k to A_l . If every inductive system in the category \mathbf{C} has an inductive limit, we say that \mathbf{C} is a category *with inductive limits*.

Theorem 2.14. Let \mathbf{C} be a category with inductive limits, and let $(\{A_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ (resp. $(\{B_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$) be an inductive system in \mathbf{C} with an inductive limit $(A, \{\phi_{k,\infty}\}_{k \in \mathbb{N}})$ (resp. $(B, \{\psi_{k,\infty}\}_{k \in \mathbb{N}})$). Let $\{\theta_k\}_{k \in \mathbb{N}}$

be a sequence of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & A_4 \xrightarrow{\phi_4} \dots \\
 \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow \\
 B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \xrightarrow{\psi_3} & B_4 \xrightarrow{\psi_4} \dots
 \end{array}$$

Then there exists a unique morphism $\theta : A \rightarrow B$ such that $\theta \circ \phi_{k,\infty} = \psi_{k,\infty} \circ \theta_k$, $k \in \mathbb{N}$.

Remark 2.15. Let \mathbf{C} be a category with inductive limits. Let $(\{A_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ be an inductive system in \mathbf{C} with inductive limit $(A, \{\phi_{k,\infty}\}_{k \in \mathbb{N}})$ and let $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ be a subsequence. Then the inductive system $(\{A_{n_k}\}_{k \in \mathbb{N}}, \{\phi_{n_k, n_{k+1}}\}_{k \in \mathbb{N}})$ has inductive limit $(A, \{\phi_{n_k, \infty}\}_{k \in \mathbb{N}})$.

Proposition 2.16. Let \mathbf{C} , $(\{A_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$, $(\{B_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$, A and B be as in Theorem 2.14. Suppose $\{\theta_{2k-1}\}_{k \in \mathbb{N}}, \{\varphi_{2k}\}_{k \in \mathbb{N}}$ are sequences of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & A_4 \xrightarrow{\phi_4} \dots \\
 \theta_1 \downarrow & & \varphi_2 \uparrow & & \theta_3 \downarrow & & \varphi_4 \uparrow \\
 B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \xrightarrow{\psi_3} & B_4 \xrightarrow{\psi_4} \dots
 \end{array}$$

Then A is isomorphic to \tilde{B} .

Remark 2.17. Let \mathbf{C} be a category with inductive limits and let $(\{A_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ (resp. $(\{B_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$) be an inductive system in \mathbf{C} with inductive limit $(A, \{\phi_{k,\infty}\}_{k \in \mathbb{N}})$ (resp. $(B, \{\psi_{k,\infty}\}_{k \in \mathbb{N}})$). By Remark 2.15 and Proposition 2.16, in order to show that A and B are isomorphic it suffices to find morphisms as in Proposition 2.16 for subsystems

$$A_{n_1} \xrightarrow{\phi_{n_1, n_2}} A_{n_2} \xrightarrow{\phi_{n_2, n_3}} A_{n_3} \xrightarrow{\phi_{n_3, n_4}} A_{n_4} \xrightarrow{\phi_{n_4, n_5}} \dots$$

and

$$B_{m_1} \xrightarrow{\psi_{m_1, m_2}} B_{m_2} \xrightarrow{\psi_{m_2, m_3}} B_{m_3} \xrightarrow{\psi_{m_3, m_4}} B_{m_4} \xrightarrow{\psi_{m_4, m_5}} \dots$$

We next recall the notion of an inverse limit in the category \mathbf{Top} whose objects are topological spaces and whose morphisms are continuous maps. Suppose we have the following inverse system in \mathbf{Top} : $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} \dots$; this means that X_k is a topological space and f_k is a continuous map, $k \in \mathbb{N}$. The inverse limit of this inverse system, denoted $\varprojlim_{\mathbf{Top}} X_k$, is the set

$$\left\{ (x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} X_k : f_k(x_{k+1}) = x_k \text{ for all } k \in \mathbb{N} \right\},$$

equipped with the product topology. We note that if each of the spaces X_k is compact and Hausdorff, then $\varprojlim_{\mathbf{Top}} X_k$ is a compact Hausdorff space.

We denote by \mathbf{C}^* the category whose objects are unital C^* -algebras and whose morphisms are unital $*$ -homomorphisms. Let

$$(3) \quad \mathcal{A}_1 \xrightarrow{\pi_1} \mathcal{A}_2 \xrightarrow{\pi_2} \mathcal{A}_3 \xrightarrow{\pi_3} \mathcal{A}_4 \xrightarrow{\pi_4} \dots$$

be an inductive system in \mathbf{C}^* . Let $\prod_{k \in \mathbb{N}} \mathcal{A}_k$ be the space of sequences $a = (a_k)_{k \in \mathbb{N}}$ such that

$$\|a\| = \sup\{\|a_k\|_{\mathcal{A}_k} : k \in \mathbb{N}\}$$

is finite. Then $\prod_{k \in \mathbb{N}} \mathcal{A}_k$, equipped with pointwise addition, multiplication and the norm $\|\cdot\|$, is a C^* -algebra. Define

$$\mathcal{A}_\infty^0 = \left\{ (a_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} \mathcal{A}_k : \exists m \in \mathbb{N} \text{ such that } \pi_k(a_k) = a_{k+1} \text{ for all } k \geq m \right\}$$

and

$$N = \left\{ (a_k)_{k \in \mathbb{N}} \in \mathcal{A}_\infty^0 : \lim_{k \rightarrow \infty} \|a_k\|_{\mathcal{A}_k} = 0 \right\}.$$

Set $\mathcal{A}_\infty = \mathcal{A}_\infty^0 / N$ and let $q : \mathcal{A}_\infty^0 \rightarrow \mathcal{A}_\infty$ be the canonical quotient map. Let $\pi_{k,\infty}^0 : \mathcal{A}_k \rightarrow \mathcal{A}_\infty^0$ be the (linear) map given by $\pi_{k,\infty}^0(a) = (b_i)_{i \in \mathbb{N}}$, where

$$b_i = \begin{cases} 0 & \text{if } i < k \\ \pi_{k,i}(a) & \text{if } i \geq k, \end{cases}$$

and let $\pi_{k,\infty} = q \circ \pi_{k,\infty}^0$. We note that $\mathcal{A}_\infty = \cup_{k \in \mathbb{N}} \pi_{k,\infty}(\mathcal{A}_k)$ and it is possible to show that $\|\pi_{k,\infty}(a_k)\|_{\mathcal{A}_\infty} = \lim_{m \rightarrow \infty} \|\pi_{k,m}(a_k)\|_{\mathcal{A}_m}$ for any $a_k \in \mathcal{A}_k$. Let $\widehat{\mathcal{A}}_\infty$ be the completion of \mathcal{A}_∞ ; then $\widehat{\mathcal{A}}_\infty$ is an inductive limit of the inductive system (3) in \mathbf{C}^* [3, Section II.8.2]. Following our general notation, we will denote it by $\varinjlim_{\mathbf{C}^*} \mathcal{A}_k$.

Remark 2.18. If each π_k is injective then $\pi_{k,\infty}$ is injective. Indeed, suppose $\pi_{k,\infty}(a_k) = 0$; then $\|a_k\|_{\mathcal{A}_k} = \lim_{m \rightarrow \infty} \|a_k\|_{\mathcal{A}_k} = \lim_{m \rightarrow \infty} \|\pi_{k,m}(a_k)\|_{\mathcal{A}_m} = 0$ and therefore $a_k = 0$.

Remark 2.19. Let $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} \dots$ be an inverse system in \mathbf{Top} such that each X_k is compact and Hausdorff. Let $C(X_1) \xrightarrow{\phi_1} C(X_2) \xrightarrow{\phi_2} C(X_3) \xrightarrow{\phi_3} C(X_4) \xrightarrow{\phi_4} \dots$ be the associated inductive system in \mathbf{C}^* . We have that $\varinjlim_{\mathbf{C}^*} C(X_i)$ is unittally $*$ -isomorphic to the C^* -algebra $C(\varprojlim_{\mathbf{Top}} X_k)$ (see [3, II.8.2.2]).

3. INDUCTIVE LIMITS OF AOU SPACES

We begin this section with the construction of the inductive limit in the category \mathbf{OU} . In Section 3.2, we identify the state space of such an inductive limit as the inverse limit of the state spaces of the intermediate ordered $*$ -vector spaces. Finally, in Section 3.3, we consider inductive limits in the category \mathbf{AOU} of AOU spaces.

3.1. Inductive limits in the category \mathbf{OU} . Let $(V_k, V_k^+, e_k)_{k \in \mathbb{N}}$ be a sequence of ordered $*$ -vector spaces with order units and let $\phi_k : V_k \rightarrow V_{k+1}$ be a unital positive map, $k \in \mathbb{N}$; thus,

$$(4) \quad V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$$

is an inductive system in \mathbf{OU} . We let

$$V_\infty^0 = \left\{ (x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} V_k : \exists m \text{ such that } \phi_k(x_k) = x_{k+1} \text{ for all } k \geq m \right\}$$

and

$$(5) \quad N_{(V_k)}^0 = \{ (x_k)_{k \in \mathbb{N}} \in V_\infty^0 : \exists m \text{ such that } x_k = 0 \text{ for all } k \geq m \}.$$

We simplify the notation and write N^0 in the place of $N_{(V_k)}^0$, when the context is clear. Clearly, N^0 is a subspace of V_∞^0 . We set

$$\ddot{V}_\infty = V_\infty^0 / N^0,$$

let $q_0 : V_\infty^0 \rightarrow \ddot{V}_\infty$ be the canonical quotient map and let $\phi_{k,\infty}^0 : V_k \rightarrow V_\infty^0$ be the (linear) map given by $\phi_{k,\infty}^0(x) = (y_i)_{i \in \mathbb{N}}$ where

$$y_i = \begin{cases} 0 & \text{if } i < k \\ \phi_{k,i}(x) & \text{if } i \geq k. \end{cases}$$

Let

$$(6) \quad \ddot{\phi}_{k,\infty} = q_0 \circ \phi_{k,\infty}^0;$$

thus, $\ddot{\phi}_{k,\infty}$ is a linear map from V_k into \ddot{V}_∞ . Since $\phi_{k,\infty}^0 = \phi_{l,\infty}^0 \circ \phi_{k,l}$, we have that

$$(7) \quad \ddot{\phi}_{k,\infty} = \ddot{\phi}_{l,\infty} \circ \phi_{k,l}, \quad k < l.$$

Note that

$$(8) \quad \ddot{V}_\infty = \bigcup_{k \in \mathbb{N}} \ddot{\phi}_{k,\infty}(V_k).$$

Remark 3.1. Let $x_k \in V_k$ and $x_l \in V_l$; then $\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{l,\infty}(x_l)$ if and only if there exists $m > \max\{k, l\}$ such that $\phi_{k,m}(x_k) = \phi_{l,m}(x_l)$.

If $x_k \in V_k$ and $x_l \in V_l$ are such that $\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{l,\infty}(x_l)$, choose $m > \max\{k, l\}$ such that $\phi_{k,m}(x_k) = \phi_{l,m}(x_l)$. Then

$$\phi_{k,m}(x_k^*) = \phi_{k,m}(x_k)^* = \phi_{l,m}(x_l)^* = \phi_{l,m}(x_l^*).$$

Therefore, $\ddot{\phi}_{k,\infty}(x_k^*) = \ddot{\phi}_{l,\infty}(x_l^*)$, and we can define an involution on \ddot{V}_∞ by letting $\ddot{\phi}_{k,\infty}(x_k)^* \stackrel{\text{def}}{=} \ddot{\phi}_{k,\infty}(x_k^*)$. It follows that $\ddot{\phi}_{k,\infty}(x_k) \in (\ddot{V}_\infty)_h$ if and only if there exists $m > k$ such that $\phi_{k,m}(x_k) \in (V_m)_h$.

Let

$$\ddot{V}_\infty^+ = \{ \ddot{\phi}_{k,\infty}(x_k) : x_k \in V_k \text{ and there exists } m \geq k \text{ with } \phi_{k,m}(x_k) \in V_m^+ \}.$$

To show that \ddot{V}_∞^+ is well-defined, suppose that $x_k \in V_k$ and $x_l \in V_l$ are such that $\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{l,\infty}(x_l)$, and that $m \geq k$ is such that $\phi_{k,m}(x_k) \in V_m^+$. Let p be such that $\phi_{k,p}(x_k) = \phi_{l,p}(x_l)$ and $q = \max\{m, p\}$. Since $\phi_{m,q}$ is positive, we have

$$\phi_{l,q}(x_l) = \phi_{k,q}(x_k) = \phi_{m,q} \circ \phi_{k,m}(x_k) \in V_q^+.$$

Lemma 3.2. *We have that*

- (i) \ddot{V}_∞^+ is a cone in $(\ddot{V}_\infty)_h$, and
- (ii) $\ddot{V}_\infty^+ \cap (-\ddot{V}_\infty^+) = \{0\}$.

Proof. (i) Let $x_k \in V_k$ be such that $\ddot{\phi}_{k,\infty}(x_k) \in \ddot{V}_\infty^+$. Then there exists $m > k$ such that $\phi_{k,m}(x_k) \in V_m^+ \subseteq (V_m)_h$, and thus $\ddot{\phi}_{k,\infty}(x_k) \in (\ddot{V}_\infty)_h$. If $r \in [0, \infty)$ then $\phi_{k,m}(rx_k) = r\phi_{k,m}(x_k) \in V_m^+$, hence $r\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{k,\infty}(rx_k) \in \ddot{V}_\infty^+$. If $\ddot{\phi}_{k,\infty}(x_k), \ddot{\phi}_{l,\infty}(x_l) \in \ddot{V}_\infty^+$ then there exist $m_1 > k$ and $m_2 > l$ such that $\phi_{k,m_1}(x_k) \in V_{m_1}^+$ and $\phi_{l,m_2}(x_l) \in V_{m_2}^+$. Set $m = \max\{m_1, m_2\}$; then $\phi_{k,m}(x_k) + \phi_{l,m}(x_l) \in V_m^+$. Therefore $\ddot{\phi}_{k,\infty}(x_k) + \ddot{\phi}_{l,\infty}(x_l) = \ddot{\phi}_{m,\infty}(\phi_{k,m}(x_k) + \phi_{l,m}(x_l)) \in \ddot{V}_\infty^+$.

(ii) Let $\ddot{\phi}_{k,\infty}(x_k) \in \ddot{V}_\infty^+ \cap (-\ddot{V}_\infty^+)$ for some $x_k \in V_k$. Then there exist $m_1, m_2 \geq k$ such that $\phi_{k,m_1}(x_k) \in V_{m_1}^+$ and $-\phi_{k,m_2}(x_k) \in V_{m_2}^+$. Choose $m > \max\{m_1, m_2\}$; then $\phi_{k,m}(x_k) \in V_m^+ \cap (-V_m^+)$, so $\phi_{k,m}(x_k) = 0$, and hence $\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{m,\infty} \circ \phi_{k,m}(x_k) = 0$. \square

Observe that $\ddot{\phi}_{k,\infty}(x_k) \leq \ddot{\phi}_{l,\infty}(x_l)$ if and only if there exists $m > \max\{k, l\}$ such that $\phi_{k,m}(x_k) \leq \phi_{l,m}(x_l)$. Furthermore, (7) implies that

$$(9) \quad \ddot{V}_\infty^+ = \bigcup_{k \in \mathbb{N}} \ddot{\phi}_{k,\infty}(V_k^+).$$

By Remark 3.1 and the unitality of the connecting maps, $\ddot{\phi}_{k,\infty}(e_k) = \ddot{\phi}_{l,\infty}(e_l)$ for all $k, l \in \mathbb{N}$. Set $\ddot{e}_\infty = \ddot{\phi}_{k,\infty}(e_k)$ (for any $k \in \mathbb{N}$). We next show that \ddot{e}_∞ is an order unit for $(\ddot{V}_\infty, \ddot{V}_\infty^+)$.

Proposition 3.3. *The triple $(\ddot{V}_\infty, \ddot{V}_\infty^+, \ddot{e}_\infty)$ is an ordered $*$ -vector space with order unit. Furthermore, $\ddot{\phi}_{k,\infty} : V_k \rightarrow \ddot{V}_\infty$ is a unital positive map such that $\ddot{\phi}_{k+1,\infty} \circ \phi_k = \ddot{\phi}_{k,\infty}$, $k \in \mathbb{N}$.*

Proof. To prove that $(\ddot{V}_\infty, \ddot{V}_\infty^+, \ddot{e}_\infty)$ is an ordered $*$ -vector space with order unit, it suffices, by Lemma 3.2, to show that \ddot{e}_∞ is an order unit. Suppose that $x_k \in V_k$ is such that $\ddot{\phi}_{k,\infty}(x_k) \in (\ddot{V}_\infty)_h$; then there exists $m > k$ such that $\phi_{k,m}(x_k) \in (V_m)_h$. Since e_m is an order unit for V_m , there exists $r_m > 0$ such that $\phi_{k,m}(x_k) \leq r_m e_m = \phi_{k,m}(r_m e_k)$. By (9),

$$\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{m,\infty} \circ \phi_{k,m}(x_k) \leq \ddot{\phi}_{m,\infty}(r_m e_m) = r_m \ddot{\phi}_{m,\infty}(e_m) = r_m \ddot{e}_\infty.$$

The identity $\ddot{\phi}_{k+1,\infty} \circ \phi_k = \ddot{\phi}_{k,\infty}$, $k \in \mathbb{N}$, is a special case of (7). \square

So far we have ascertained that $(\ddot{V}_\infty, \{\ddot{\phi}_{k,\infty}\}_{k \in \mathbb{N}})$ is a suitable candidate for the inductive limit in **OU** of the inductive system (4). Theorem 3.5

will verify that this pair does indeed satisfy the universal property of the inductive limit. First we take note of the special case when the maps in the inductive system are unital order isomorphisms. Since the proof of the statement is straightforward, we omit it.

Remark 3.4. Let $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **OU** such that ϕ_k is an order isomorphism onto its image for all $k \in \mathbb{N}$. Then $\check{\phi}_{k,\infty}$ is an order isomorphism onto its image for all $k \in \mathbb{N}$.

Theorem 3.5. The triple $(\check{V}_\infty, \{\check{\phi}_{k,\infty}\}_{k \in \mathbb{N}}, \check{e}_\infty)$ is an inductive limit of the inductive system $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ in **OU**.

Proof. We check that $(\check{V}_\infty, \{\check{\phi}_{k,\infty}\}_{k \in \mathbb{N}})$ satisfies the universal property of the inductive limit. Suppose $(W, \{\psi_k\}_{k \in \mathbb{N}})$ is a pair consisting of an ordered $*$ -vector space and a family of unital positive maps $\psi_k : V_k \rightarrow W$ such that $\psi_{k+1} \circ \phi_k = \psi_k$ for all $k \in \mathbb{N}$. Let $k, l \in \mathbb{N}$, $x_k \in V_k$, $x_l \in V_l$ and suppose that $\check{\phi}_{k,\infty}(x_k) = \check{\phi}_{l,\infty}(x_l)$. By Remark 3.1, there exists $m > \max\{k, l\}$ such that $\phi_{k,m}(x_k) = \phi_{l,m}(x_l)$. Consequently $\psi_k(x_k) = \psi_m \circ \phi_{k,m}(x_k) = \psi_m \circ \phi_{l,m}(x_l) = \psi_l(x_l)$. Let $\check{\psi} : \check{V}_\infty \rightarrow W$ be given by $\check{\psi} \circ \check{\phi}_{k,\infty} = \psi_k$; since $\check{V}_\infty = \cup_{k \in \mathbb{N}} \check{\phi}_{k,\infty}(V_k)$, the map $\check{\psi}$ is well-defined. Since ψ_k is unital and $\check{\psi} \circ \check{\phi}_{k,\infty}(e_k) = \psi_k(e_k)$, the map $\check{\psi}$ is unital. Suppose that $\check{\phi}_{k,\infty}(x_k) \in \check{V}_\infty^+$; then there exists $m > k$ such that $\phi_{k,m}(x_k) \in V_m^+$. Since ψ_m is positive and $\check{\psi} \circ \check{\phi}_{k,\infty}(x_k) = \psi_k(x_k) = \psi_m \circ \phi_{k,m}(x_k)$, we have that $\check{\psi}(\check{\phi}_{k,\infty}(x_k)) \in W^+$ and hence $\check{\psi}$ is positive. \square

According to our general notation, denote by $\varinjlim_{\mathbf{OU}} V_k$ the inductive limit $(\check{V}_\infty, \{\check{\phi}_{k,\infty}\}_{k \in \mathbb{N}})$.

Remark 3.6. Let $(\{V_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ and $(\{W_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$ be inductive systems in **OU** and let $\{\theta_k\}_{k \in \mathbb{N}}$ be a sequence of unital positive maps such that the following diagram commutes:

$$(10) \quad \begin{array}{ccccccc} V_1 & \xrightarrow{\phi_1} & V_2 & \xrightarrow{\phi_2} & V_3 & \xrightarrow{\phi_3} & V_4 & \xrightarrow{\phi_4} & \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow & & \\ W_1 & \xrightarrow{\psi_1} & W_2 & \xrightarrow{\psi_2} & W_3 & \xrightarrow{\psi_3} & W_4 & \xrightarrow{\psi_4} & \dots \end{array}$$

It follows from Theorem 3.5 and Theorem 2.14 that there exists a unique unital positive map $\check{\theta} : \varinjlim_{\mathbf{OU}} V_k \rightarrow \varinjlim_{\mathbf{OU}} W_k$ such that $\check{\theta} \circ \check{\phi}_{k,\infty} = \check{\psi}_{k,\infty} \circ \theta_k$ for all $k \in \mathbb{N}$.

- (i) If θ_k is injective for every $k \in \mathbb{N}$ then $\check{\theta}$ is injective. Indeed, if $x_k \in V_k$ and $\check{\theta} \circ \check{\phi}_{k,\infty}(x_k) = 0$, then $\check{\psi}_{k,\infty} \circ \theta_k(x_k) = 0$. Therefore there exists $m > \max\{k, l\}$ such that $\psi_{k,m} \circ \theta_k(x_k) = 0$. Since (10) commutes, $\theta_m \circ \phi_{k,m}(x_k) = 0$. Since θ_m is injective, $\phi_{k,m}(x_k) = 0$ and hence $\check{\phi}_{k,\infty}(x_k) = 0$.

- (ii) If θ_k is an order isomorphism onto its image for every $k \in \mathbb{N}$ then $\ddot{\theta}$ is an order isomorphism onto its image. Indeed, suppose that $\ddot{\theta} \circ \ddot{\phi}_{k,\infty}(x_k) \in (\varinjlim_{\mathbf{OU}} W_k)^+$ for some $x_k \in V_k$. Then $\ddot{\psi}_{k,\infty} \circ \theta_k(x_k) \in (\varinjlim_{\mathbf{OU}} W_k)^+$ and it follows that there exists $m > k$ such that $\psi_{k,m} \circ \theta_k(x_k) \in W_m^+$. Since (10) commutes, this implies that $\theta_m \circ \phi_{k,m}(x_k) \in W_m^+$. Since θ_m is an order isomorphism, it follows that $\phi_{k,m}(x_k) \in V_m^+$, and hence $\ddot{\phi}_{k,\infty}(x_k) \in (\varinjlim_{\mathbf{OU}} V_k)^+$.

3.2. The state space of the inductive limit in \mathbf{OU} . Given the inductive system (4), one can “reverse the arrows” to obtain a sequence

$$V'_1 \xleftarrow{\phi'_1} V'_2 \xleftarrow{\phi'_2} V'_3 \xleftarrow{\phi'_3} V'_4 \xleftarrow{\phi'_4} \dots$$

of dual spaces and continuous maps (here we use the fact that unital positive maps between \mathbf{OU} spaces are automatically continuous in the order norm [33, Theorem 4.22]). Since the maps ϕ_k are unital, we have that $\phi'_k(S(V_{k+1})) \subseteq S(V_k)$ for all $k \in \mathbb{N}$, and thus we obtain the following inverse system in **Top**:

$$(11) \quad S(V_1) \xleftarrow{\phi'_1} S(V_2) \xleftarrow{\phi'_2} S(V_3) \xleftarrow{\phi'_3} S(V_4) \xleftarrow{\phi'_4} \dots$$

Proposition 3.7. *Let $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OU} . The state space $S(\varinjlim_{\mathbf{OU}} V_k)$ is topologically homeomorphic to the inverse limit $\varprojlim_{\mathbf{Top}} S(V_k)$.*

Proof. Let $f \in S(\varinjlim_{\mathbf{OU}} V_k)$ and define $f_k : V_k \rightarrow \mathbb{C}$ by letting $\langle f_k, x_k \rangle = \langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle$, $x_k \in V_k$. For $x_k \in V_k$, we have

$$\begin{aligned} \langle f_{k+1} \circ \phi_k, x_k \rangle &= \langle f_{k+1}, \phi_k(x_k) \rangle = \langle f, \ddot{\phi}_{k+1,\infty} \circ \phi_k(x_k) \rangle \\ &= \langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k, x_k \rangle. \end{aligned}$$

Therefore $\phi'_k(f_{k+1}) = f_k$ and so $(f_k)_{k \in \mathbb{N}} \in \varprojlim_{\mathbf{Top}} S(V_k)$. Define a map $\theta : S(\varinjlim_{\mathbf{OU}} V_k) \rightarrow \varprojlim_{\mathbf{Top}} S(V_k)$ by letting $\theta(f) = (f_k)_{k \in \mathbb{N}}$.

Suppose $f, g \in S(\varinjlim_{\mathbf{OU}} V_k)$ are such that $\theta(f) = \theta(g)$; that is, $f_k = g_k$ for all $k \in \mathbb{N}$. If $x_k \in V_k$ then

$$\langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k, x_k \rangle = \langle g_k, x_k \rangle = \langle g, \ddot{\phi}_{k,\infty}(x_k) \rangle.$$

By (8), $f = g$ and hence θ is injective.

Given a sequence $(f_k)_{k \in \mathbb{N}} \in \varprojlim_{\mathbf{Top}} S(V_k)$, define an element $f : \ddot{V}_\infty \rightarrow \mathbb{C}$ by setting $\langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k, x_k \rangle$, $x_k \in V_k$. Observe that f is well-defined, for if $\ddot{\phi}_{k,\infty}(x_k) = \ddot{\phi}_{l,\infty}(x_l)$ for some $x_k \in V_k$ and $x_l \in V_l$ then, by Remark 3.1, there exists $m > \max\{k, l\}$ such that $\phi_{k,m}(x_k) = \phi_{l,m}(x_l)$. Hence

$$\begin{aligned} \langle f_k, x_k \rangle &= \langle f_m \circ \phi_{k,m}, x_k \rangle = \langle f_m, \phi_{k,m}(x_k) \rangle \\ &= \langle f_m, \phi_{l,m}(x_l) \rangle = \langle f_m \circ \phi_{l,m}, x_l \rangle = \langle f_l, x_l \rangle. \end{aligned}$$

Suppose that $x \in (\varinjlim_{\mathbf{OU}} V_k)^+$. By (9), there exist $k \in \mathbb{N}$ and $x_k \in V_k^+$ such that $x = \ddot{\phi}_{k,\infty}(x_k)$, and hence

$$\langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k, x_k \rangle \geq 0,$$

showing that f is positive. Furthermore, $\langle f, \ddot{e}_\infty \rangle = \langle f_k, e_k \rangle = 1$ and thus $f \in S(\varinjlim_{\mathbf{OU}} V_k)$. Since $\theta(f) = (f_k)_{k \in \mathbb{N}}$, we conclude that θ is surjective.

Finally, we prove that θ is a homeomorphism. Suppose that $(f^\lambda)_{\lambda \in \Lambda} \in S(\varinjlim_{\mathbf{OU}} V_k)$ is a net such that $f^\lambda \rightarrow_{\lambda \in \Lambda} f$ for some $f \in S(\varinjlim_{\mathbf{OU}} V_k)$. Write

$$\theta(f) = (f_k)_{k \in \mathbb{N}} \text{ and } \theta(f^\lambda) = (f_k^\lambda)_{k \in \mathbb{N}}, \quad \lambda \in \Lambda.$$

Since $\varprojlim_{\mathbf{Top}} S(V_k)$ is equipped with the product topology,

$$((f_k^\lambda)_{k \in \mathbb{N}})_{\lambda \in \Lambda} \rightarrow_{\lambda \in \Lambda} (f_k)_{k \in \mathbb{N}} \text{ if and only if } (f_k^\lambda)_{\lambda \in \Lambda} \rightarrow_{\lambda \in \Lambda} f_k \text{ for all } k \in \mathbb{N}.$$

If $k \in \mathbb{N}$ and $x_k \in V_k$ then

$$\langle f_k^\lambda, x_k \rangle = \langle f^\lambda, \ddot{\phi}_{k,\infty}(x_k) \rangle \rightarrow_{\lambda \in \Lambda} \langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k, x_k \rangle.$$

It follows that $\theta(f^\lambda) \rightarrow_{\lambda \in \Lambda} \theta(f)$ and so θ is continuous.

Suppose that $((f_k^\lambda)_{k \in \mathbb{N}})_{\lambda \in \Lambda} \in \varprojlim_{\mathbf{Top}} S(V_k)$ is such that $(f_k^\lambda)_{k \in \mathbb{N}} \rightarrow_{\lambda \in \Lambda} (f_k)_{k \in \mathbb{N}}$. For each $k \in \mathbb{N}$, $(f_k^\lambda)_{\lambda \in \Lambda} \rightarrow_{\lambda \in \Lambda} f_k$. Now,

$$\theta^{-1}((f_k^\lambda)_{k \in \mathbb{N}}) = f^\lambda, \text{ where } \langle f^\lambda, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k^\lambda, x_k \rangle.$$

If $x_k \in V_k$ then

$$\langle f^\lambda, \ddot{\phi}_{k,\infty}(x_k) \rangle = \langle f_k^\lambda, x_k \rangle \rightarrow_{\lambda \in \Lambda} \langle f_k, x_k \rangle = \langle f, \ddot{\phi}_{k,\infty}(x_k) \rangle.$$

By (8), $\theta^{-1}((f_k^\lambda)_{k \in \mathbb{N}}) \rightarrow_{\lambda \in \Lambda} \theta^{-1}((f_k)_{k \in \mathbb{N}})$ and therefore θ is a homeomorphism. \square

3.3. Inductive limits in the category \mathbf{AOU} . Let $(V_k, V_k^+, e_k)_{k \in \mathbb{N}}$ be a sequence of Archimedean order unit spaces and

$$(12) \quad V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$$

be an inductive system in the category \mathbf{AOU} . Recall that this means that $\phi_k : V_k \rightarrow V_{k+1}$ is a unital positive map, $k \in \mathbb{N}$.

The proof of the following remark is straightforward and we omit it.

Remark 3.8. Let $\|\cdot\|^k$ be an order norm on V_k , $k \in \mathbb{N}$. For $x_k \in V_k$, we have that $\lim_{m \rightarrow \infty} \|\phi_{k,m}(x_k)\|^m = 0$ if and only if

$$\lim_{m \rightarrow \infty} \|\operatorname{Re}(\phi_{k,m}(x_k))\|_h = 0 \text{ and } \lim_{m \rightarrow \infty} \|\operatorname{Im}(\phi_{k,m}(x_k))\|_h = 0.$$

Let $\|\cdot\|^k$ be any order norm on V_k and $\|\cdot\|^\infty$ be any order seminorm on $\varinjlim_{\mathbf{OU}} V_k$. Let

$$(13) \quad N = \{x \in \varinjlim_{\mathbf{OU}} V_k : \|x\|^\infty = 0\}$$

be the kernel of $\|\cdot\|^\infty$.

Proposition 3.9. *Let $x_k \in V_k$ and $x = \ddot{\phi}_{k,\infty}(x_k) \in \varinjlim_{\mathbf{OU}} V_k$. The following are equivalent:*

- (i) $x \in N$;
- (ii) $\lim_{m \rightarrow \infty} \|\phi_{k,m}(x_k)\|^m = 0$.

Proof. By Remarks 2.1 and 3.8, we may assume that $x \in (\varinjlim_{\mathbf{OU}} V_k)_h$.

(i) \Rightarrow (ii) We have that

$$\inf\{\lambda \geq 0 : -\lambda \ddot{e}_\infty \leq \ddot{\phi}_{k,\infty}(x_k) \leq \lambda \ddot{e}_\infty\} = 0.$$

Let $r > 0$; then there exists $m \in \mathbb{N}$ such that $-re_l \leq \phi_{k,l}(x_k) \leq re_l$ for all $l \geq m$. Therefore $\|\phi_{k,l}(x_k)\|^l \leq r$ for all $l \geq m$. Thus, $\lim_{m \rightarrow \infty} \|\phi_{k,m}(x_k)\|^m = 0$.

(ii) \Rightarrow (i) Assume, towards a contradiction, that $\|x\|^\infty = \mu > 0$. There exists $m > k$ such that

$$\inf\{\lambda_l : -\lambda_l e_l \leq \phi_{k,l}(x_k) \leq \lambda_l e_l\} < \frac{\mu}{2}, \quad l \geq m.$$

Therefore, $-\frac{\mu}{2}e_l \leq \phi_{k,l}(x_k) \leq \frac{\mu}{2}e_l$ for all $l \geq m$ and so $-\frac{\mu}{2}\ddot{\phi}_{k,\infty}(e_k) \leq \ddot{\phi}_{k,\infty}(x_k) \leq \frac{\mu}{2}\ddot{\phi}_{k,\infty}(e_k)$. Thus $\|x\|^\infty \leq \frac{\mu}{2} < \mu$, a contradiction. \square

In view of Proposition 3.9, we will refer to N defined by (13) as the *null space of the sequence* $(V_k, V_k^+, e_k)_{k \in \mathbb{N}}$.

We may apply the forgetful functor $\mathbf{F} : \mathbf{AOU} \rightarrow \mathbf{OU}$ and consider the inductive limit $\varinjlim_{\mathbf{OU}} \mathbf{F}(V_k)$. This is not necessarily an AOUspace. Indeed, let $W_n = \ell^\infty(B_n)$, where $B_n = \{k \in \mathbb{N} : k \geq n\}$, and $\psi_n : W_n \rightarrow W_{n+1}$ be the restriction map, $\psi_n(f) = f|_{B_{n+1}}$. Note that $W_1 = \ell^\infty$. Let $f \in c_0$ have strictly positive entries. Then, by Proposition 3.9, $\ddot{\phi}_{1,\infty}(f) \in N$; by [33, Proposition 2.23], the unit of $\varinjlim_{\mathbf{OU}} \mathbf{F}(W_k)$ is not Archimedean. We shall now however see that the Archimedeanisation of the OU space $\varinjlim_{\mathbf{OU}} \mathbf{F}(V_k)$ is always an inductive limit in \mathbf{AOU} .

Let $(V_\infty, V_\infty^+, e_\infty)$ be the Archimedeanisation of $\varinjlim_{\mathbf{OU}} V_k$; thus,

$$V_\infty = (\varinjlim_{\mathbf{OU}} V_k)/N,$$

the involution on V_∞ is given by $(\ddot{\phi}_{k,\infty}(x_k) + N)^* = \ddot{\phi}_{k,\infty}(x_k)^* + N$ (for $x_k \in V_k$),

$$V_\infty^+ = \{\ddot{\phi}_{k,\infty}(x_k) + N : x_k \in (V_k)_h, k \in \mathbb{N}, \text{ and}$$

$$\ddot{\phi}_{k,\infty}(x_k) + r\ddot{\phi}_{k,\infty}(e_k) \in \ddot{V}_\infty^+, \text{ for all } r > 0\},$$

and $e_\infty = \ddot{e}_\infty + N$.

Lemma 3.10. *Let $x_k \in V_k$. The following are equivalent:*

- (i) $\ddot{\phi}_{k,\infty}(x_k) + N \in (V_\infty)_h$;
- (ii) $\ddot{\phi}_{k,\infty}(x_k) + N = \ddot{\phi}_{k,\infty}(\text{Re}(x_k)) + N$;
- (iii) $\ddot{\phi}_{k,\infty}(x_k) + N = \ddot{\phi}_{l,\infty}(x_l) + N$ for some $l \in \mathbb{N}$ and some $x_l \in (V_l)_h$.

Proof. (i) \Rightarrow (ii) Suppose $\ddot{\phi}_{k,\infty}(x_k) + N \in (V_\infty)_h$. Then $\ddot{\phi}_{k,\infty}(x_k) + N = \ddot{\phi}_{k,\infty}(x_k)^* + N = \ddot{\phi}_{k,\infty}(x_k^*) + N$ and therefore

$$\begin{aligned} \ddot{\phi}_{k,\infty}(x_k) + N &= \frac{\ddot{\phi}_{k,\infty}(x_k) + \ddot{\phi}_{k,\infty}(x_k^*)}{2} + N \\ &= \ddot{\phi}_{k,\infty}\left(\frac{x_k + x_k^*}{2}\right) + N = \ddot{\phi}_{k,\infty}(\operatorname{Re}(x_k)) + N. \end{aligned}$$

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Suppose $\ddot{\phi}_{k,\infty}(x_k) + N = \ddot{\phi}_{l,\infty}(x_l) + N$ for some $x_l \in (V_l)_h$. Then

$$\begin{aligned} (\ddot{\phi}_{k,\infty}(x_k) + N)^* &= (\ddot{\phi}_{l,\infty}(x_l) + N)^* = \ddot{\phi}_{l,\infty}(x_l)^* + N \\ &= \ddot{\phi}_{l,\infty}(x_l^*) + N = \ddot{\phi}_{l,\infty}(x_l) + N = \ddot{\phi}_{k,\infty}(x_k) + N. \end{aligned}$$

□

Remark 3.11. We have that

$$V_\infty^+ = \{\ddot{\phi}_{k,\infty}(x_k) + N : x_k \in (V_k)_h, k \in \mathbb{N}, \text{ and}$$

for every $r > 0$ there exists $m \geq k$ such that $\phi_{k,m}(x_k) + re_m \in V_m^+\}$.

An element $\ddot{\phi}_{k,\infty}(x_k) + N \in (V_\infty)_h$ (where $x_k \in (V_k)_h$) belongs to V_∞^+ if and only if for every $r > 0$ there exist $l \in \mathbb{N}$ and $y_l \in V_l$ such that $\ddot{\phi}_{l,\infty}(y_l) \in N$ and $\ddot{\phi}_{k,\infty}(re_k + x_k) + \ddot{\phi}_{l,\infty}(y_l) \in \ddot{V}_\infty^+$. Thus, $\ddot{\phi}_{k,\infty}(x_k) + N \in V_\infty^+$ if and only if for every $r > 0$ there exist $l \in \mathbb{N}$ and $y_l \in V_l$ such that $\ddot{\phi}_{l,\infty}(y_l) \in N$, and there exists $m > \max\{k, l\}$ with $re_m + \phi_{k,m}(x_k) + \phi_{l,m}(y_l) \in V_m^+$. We may assume without loss of generality that $l > k$ and that $y_l \in (V_l)_h$.

Let $q_V : \ddot{V}_\infty \rightarrow V_\infty$ be the canonical quotient map, and set

$$\phi_{k,\infty} = q_V \circ \ddot{\phi}_{k,\infty};$$

we have that $\phi_{k,\infty}$ is a unital positive map and

$$\phi_{k+1,\infty} \circ \phi_k = \phi_{k,\infty}, \quad k \in \mathbb{N}.$$

Since $\ddot{V}_\infty = \cup_{k \in \mathbb{N}} \ddot{\phi}_{k,\infty}(V_k)$, we have that

$$V_\infty = \cup_{k \in \mathbb{N}} \phi_{k,\infty}(V_k).$$

The following lemma is certainly well-known; we record it since we were not able to find a precise reference.

Lemma 3.12. *Let (V, V^+, e) be an AOU space and $W \subseteq V$ be a linear $*$ -subspace containing e . Set $W^+ = W \cap V^+$. Then (W, W^+, e) is an AOU space and for every $f \in S(W)$ there exists $g \in S(V)$ such that $g|_W = f$.*

Proof. It is straightforward to check that (W, W^+, e) is an AOU space. Recall the correspondence between complex functionals on V and real functionals on V_h : given a real functional ω on V_h , one defines a functional $\tilde{\omega} : V \rightarrow \mathbb{C}$ by letting $\tilde{\omega}(x) = \omega(\operatorname{Re}(x)) + i\omega(\operatorname{Im}(x))$, $x \in V$. The second statement now follows from the fact that, by [33, Proposition 3.11], ω is positive if and only if $\tilde{\omega}$ is positive, and by [33, Corollary 2.15], every positive real functional on

a real ordered vector space can be extended to a positive real functional on a larger space. \square

Proposition 3.13. *Let $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **AOU** such that ϕ_k is an order isomorphism onto its image for each $k \in \mathbb{N}$. Then $N = \{0\}$ and $\phi_{k,\infty}$ is a unital order isomorphism onto its image for all $k \in \mathbb{N}$.*

Proof. Suppose that $x_k \in V_k$ and $\ddot{\phi}_{k,\infty}(x_k) \in N$. By Proposition 3.9, $\lim_{m \rightarrow \infty} \|\phi_{k,m}(x_k)\| = 0$. Since each ϕ_k is an order isomorphism onto $\phi_k(V_k)$, using Lemma 3.12 we obtain that $\|\phi_{k,m}(x_k)\| = \|x_k\|$ for all $m \geq k$ and so $x_k = 0$. Thus, $\ddot{\phi}_{k,\infty}(x_k) = 0$. It now follows that $\phi_{k,\infty} = \ddot{\phi}_{k,\infty}$ and therefore, by Remark 3.4, $\phi_{k,\infty}$ is a unital order isomorphism onto its image, $k \in \mathbb{N}$. \square

Theorem 3.14. *The triple $(V_\infty, \{\phi_{k,\infty}\}_{k \in \mathbb{N}}, e_\infty)$ is the inductive limit of the inductive system $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ in the category **AOU**.*

Proof. Suppose $(W, \{\psi_k\}_{k \in \mathbb{N}})$ is a pair consisting of an AOU space and a family of unital positive maps $\psi_k : V_k \rightarrow W$ such that $\psi_{k+1} \circ \phi_k = \psi_k$ for all $k \in \mathbb{N}$. By Theorem 3.5, there exists a unique unital positive map $\ddot{\psi} : \varinjlim_{\mathbf{AOU}} V_k \rightarrow W$ such that $\ddot{\psi} \circ \ddot{\phi}_{k,\infty} = \psi_k$ for all $k \in \mathbb{N}$. By Theorem 2.2, there exists a unique unital positive map $\psi : V_\infty \rightarrow W$ such that $\psi \circ q_V = \ddot{\psi}$. Therefore $\psi \circ \phi_{k,\infty} = \psi \circ q_V \circ \ddot{\phi}_{k,\infty} = \ddot{\psi} \circ \ddot{\phi}_{k,\infty} = \psi_k$ for all $k \in \mathbb{N}$ and the proof is complete. \square

We recall that, according to our general notation for inductive limits, $\varinjlim_{\mathbf{AOU}} V_k$ will henceforth stand for the AOU space $(V_\infty, \{\phi_{k,\infty}\}_{k \in \mathbb{N}}, e_\infty)$.

Remark 3.15. For each $k \in \mathbb{N}$, let (V_k, V_k^+, e_k) and (W_k, W_k^+, f_k) be AOU spaces such that $(\{V_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ and $(\{W_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$ are inductive systems and let $\{\theta_k\}_{k \in \mathbb{N}}$ be a sequence of unital positive maps such that the following diagram commutes:

$$(14) \quad \begin{array}{ccccccc} V_1 & \xrightarrow{\phi_1} & V_2 & \xrightarrow{\phi_2} & V_3 & \xrightarrow{\phi_3} & V_4 & \xrightarrow{\phi_4} & \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow & & \\ W_1 & \xrightarrow{\psi_1} & W_2 & \xrightarrow{\psi_2} & W_3 & \xrightarrow{\psi_3} & W_4 & \xrightarrow{\psi_4} & \dots \end{array}$$

It follows from Theorem 3.14 and Theorem 2.14 that there exists a unique unital positive map $\theta : \varinjlim_{\mathbf{AOU}} V_k \rightarrow \varinjlim_{\mathbf{AOU}} W_k$ such that $\theta \circ \phi_{k,\infty} = \psi_{k,\infty} \circ \theta_k$ for all $k \in \mathbb{N}$. It is easy to see that if θ_k is an order isomorphism onto its image for each $k \in \mathbb{N}$ then θ is injective.

Proposition 3.16. *Let $(\{V_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ be an inductive system in **AOU**. Then $S(\varinjlim_{\mathbf{AOU}} V_k)$ is homeomorphic to $\varinjlim_{\mathbf{Top}} S(V_k)$.*

Proof. If $f \in S(\varinjlim_{\mathbf{OU}} V_k)$ then, by Theorem 2.2, there exists a unique unital positive map $\tilde{f} \in S(\varinjlim_{\mathbf{AOU}} V_k)$ such that $f = \tilde{f} \circ q_V$. Define $\theta : S(\varinjlim_{\mathbf{OU}} V_k) \rightarrow S(\varinjlim_{\mathbf{AOU}} V_k)$ by letting $\theta(f) = \tilde{f}$; it is straightforward to check that θ is a homeomorphism (recall that the state space is equipped with the weak* topology). By Proposition 3.7, $S(\varinjlim_{\mathbf{OU}} V_k)$ is homeomorphic to $\varprojlim_{\mathbf{Top}} S(V_k)$, and the claim follows. \square

4. INDUCTIVE LIMITS OF OPERATOR SYSTEMS

We begin this section with the construction of the inductive limit in the category **MOU** of matrix ordered spaces, and in Section 4.2 we consider the inductive limit in the category **OS** of operator systems. We devote the remainder of the chapter to proving various “commutation theorems” for the inductive limit in **OS**. In particular, we prove that the inductive limit intertwines OMAX and commutes with the maximal operator system tensor product. Analogous results hold for OMIN and the minimal operator system tensor product, provided the connecting morphisms are complete order embeddings. We note that the commutation with the minimal tensor product in the case of complete operator systems was recently proved in [24]. We also establish, under certain natural conditions, the commutation of the inductive limit with the quotient construction.

4.1. Inductive limits of matrix ordered *-vector spaces. In this subsection, let $(\mathcal{S}_k, \{C_n^k\}_{n \in \mathbb{N}}, e_k)_{k \in \mathbb{N}}$ be a sequence of matrix ordered *-vector spaces with matrix order unit and $\phi_k : \mathcal{S}_k \rightarrow \mathcal{S}_{k+1}$ be a unital completely positive map, $k \in \mathbb{N}$; thus,

$$(15) \quad \mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$$

is an inductive system in **MOU**. For each $n \in \mathbb{N}$, consider the induced inductive system in **OU**:

$$M_n(\mathcal{S}_1) \xrightarrow{\phi_1^{(n)}} M_n(\mathcal{S}_2) \xrightarrow{\phi_2^{(n)}} M_n(\mathcal{S}_3) \xrightarrow{\phi_3^{(n)}} M_n(\mathcal{S}_4) \xrightarrow{\phi_4^{(n)}} \dots$$

Denote by $\check{\phi}_{k,\infty}^n$ the unital positive map associated to $\varinjlim_{\mathbf{OU}} M_n(\mathcal{S}_k)$ through (6), so that $\check{\phi}_{k,\infty}^n : M_n(\mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{OU}} M_n(\mathcal{S}_k)$ and $\check{\phi}_{k+1,\infty}^n \circ \phi_k^{(n)} = \check{\phi}_{k,\infty}^n$ for all $k \in \mathbb{N}$. Note that $\check{\phi}_{k,\infty}^1 = \check{\phi}_{k,\infty}$. We caution the reader about the difference between the maps $\check{\phi}_{k,\infty}^n$ and $\check{\phi}_{k,\infty}^{(n)}$: while their domains are both equal to $M_n(\mathcal{S})$, their ranges are within $\varinjlim_{\mathbf{OU}} M_n(\mathcal{S}_k)$ and $M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k)$, respectively.

Lemma 4.1. *We have that $M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k) = \bigcup_{k \in \mathbb{N}} \check{\phi}_{k,\infty}^{(n)} M_n(\mathcal{S}_k)$, $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$. It is clear that $\check{\phi}_{k,\infty}^{(n)}(M_n(\mathcal{S}_k)) \subseteq M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k)$ for all k . To show the reverse inclusion, let $(s_{i,j})_{i,j} \in M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k)$. For all $1 \leq i, j \leq n$, we have that $s_{i,j} = \check{\phi}_{k_{i,j},\infty}^{(n)}(s_{k_{i,j}})$ for some $k_{i,j} \in \mathbb{N}$ and $s_{k_{i,j}} \in \mathcal{S}_{k_{i,j}}$.

Let $k = \max\{k_{i,j} : 1 \leq i, j \leq n\}$ and $s_{i,j}^k = \phi_{k_{i,j},k}(s_{k_{i,j}})$. We have that $s_{i,j} = \ddot{\phi}_{k,\infty}(s_{i,j}^k)$ for all $1 \leq i, j \leq n$ and hence $(s_{i,j})_{i,j} \in \ddot{\phi}_{k,\infty}^{(n)}(M_n(\mathcal{S}_k))$. \square

In the next lemma, $M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k)$ is equipped with its canonical involution arising from the involution of $\varinjlim_{\mathbf{OU}} \mathcal{S}_k$.

Lemma 4.2. *The mapping $\pi_n : M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{OU}} M_n(\mathcal{S}_k)$ given by*

$$\pi_n \circ \ddot{\phi}_{k,\infty}^{(n)} = \ddot{\phi}_{k,\infty}^n, \quad k \in \mathbb{N},$$

is well-defined, bijective and involutive.

Proof. Fix $n \in \mathbb{N}$ and let $S \in M_n(\varinjlim_{\mathbf{OU}} \mathcal{S}_k)$. By Lemma 4.1, $S = \ddot{\phi}_{k,\infty}^{(n)}(S_k)$ for some $k \in \mathbb{N}$ and some $S_k \in M_n(\mathcal{S}_k)$. Suppose that $S_k = (s_{i,j}^k)_{i,j} \in M_n(\mathcal{S}_k)$ and $S_l = (s_{i,j}^l)_{i,j} \in M_n(\mathcal{S}_l)$ are such that $\ddot{\phi}_{k,\infty}^{(n)}(S_k) = \ddot{\phi}_{l,\infty}^{(n)}(S_l)$; then, for all $1 \leq i, j \leq n$, there exists $m_{i,j}$ such that $\phi_{k,m_{i,j}}(s_{i,j}^k) = \phi_{l,m_{i,j}}(s_{i,j}^l)$. Let $m = \max\{m_{i,j} : 1 \leq i, j \leq n\}$; we have $\phi_{k,m}^{(n)}(S_k) = \phi_{l,m}^{(n)}(S_l)$. Therefore $\ddot{\phi}_{k,\infty}^n(S_k) = \ddot{\phi}_{l,\infty}^n(S_l)$. It follows that the mapping π_n is well-defined. Since the mappings $\ddot{\phi}_{k,\infty}^{(n)}$ and $\ddot{\phi}_{k,\infty}^n$ are linear, we have that π_n is linear.

Suppose $S_k \in M_n(\mathcal{S}_k)$ is such that $\ddot{\phi}_{k,\infty}^n(S_k) = 0$. Then there exists $m > k$ such that $\phi_{k,m}^{(n)}(S_k) = 0$ and therefore $\ddot{\phi}_{k,\infty}^{(n)}(S_k) = \ddot{\phi}_{m,\infty}^{(n)} \circ \phi_{k,m}^{(n)}(S_k) = 0$. This shows that π_n is injective. The verification that π_n is involutive is straightforward and is omitted. \square

We denote $\varinjlim_{\mathbf{OU}} \mathcal{S}_k$ by $\ddot{\mathcal{S}}_\infty$ and let, as before, $\ddot{e}_\infty = \ddot{\phi}_{k,\infty}(e_k)$ for any $k \in \mathbb{N}$ (note that \ddot{e}_∞ is thus well-defined). For each $n \in \mathbb{N}$, let $C_n \subseteq M_n(\ddot{\mathcal{S}}_\infty)$ be given by

$$C_n = \pi_n^{-1}((\varinjlim_{\mathbf{OU}} M_n(\mathcal{S}_k))^+).$$

Proposition 4.3. *The triple $(\ddot{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \ddot{e}_\infty)$ is a matrix ordered $*$ -vector space with matrix order unit.*

Proof. Since C_n is the inverse image of a proper cone under the injective mapping π_n (Lemma 4.2), we have that C_n is a proper cone itself. We show that the family $\{C_n\}_{n \in \mathbb{N}}$ is compatible. Let $n, m \in \mathbb{N}$, $\alpha \in M_{n,m}$ and $\ddot{\phi}_{k,\infty}^{(n)}(S_k) \in C_n$, where $S_k \in M_n(\mathcal{S}_k)$. There exists $p \in \mathbb{N}$ such that $\phi_{k,p}^{(n)}(S_k) \in M_n(\mathcal{S}_p)^+$. We conclude that $\alpha^* \phi_{k,p}^{(n)}(S_k) \alpha \in M_m(\mathcal{S}_p)^+$ and so $\ddot{\phi}_{p,\infty}^m(\alpha^* \phi_{k,p}^{(n)}(S_k) \alpha) \in (\varinjlim_{\mathbf{OU}} M_m(\mathcal{S}_k))^+$. Therefore

$$\alpha^* \ddot{\phi}_{k,\infty}^{(n)}(S_k) \alpha = \ddot{\phi}_{p,\infty}^m(\alpha^* \phi_{k,p}^{(n)}(S_k) \alpha) \in C_m.$$

Thus, $\{C_n\}_{n \in \mathbb{N}}$ is a matrix ordering for $\ddot{\mathcal{S}}_\infty$. The fact that \ddot{e}_∞ is a matrix order unit can be shown as in Proposition 3.3, and detailed the proof is omitted. \square

For the remainder of this section, we denote by $\ddot{\mathcal{S}}_\infty$ the matrix ordered *-vector space with matrix order unit $(\ddot{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \ddot{e}_\infty)$.

Remark 4.4. The map $\ddot{\phi}_{k,\infty} : \mathcal{S}_k \rightarrow \ddot{\mathcal{S}}_\infty$ is unital and completely positive. Indeed, suppose $S_k \in M_n(\mathcal{S}_k)^+$. Since $\ddot{\phi}_{k,\infty}^n$ is a unital positive map, $\ddot{\phi}_{k,\infty}^n(S_k) \in (\varinjlim_{\mathbf{MOU}} M_n(\mathcal{S}_k))^+$ and therefore $\ddot{\phi}_{k,\infty}^{(n)}(S_k) \in C_n$.

Proposition 4.5. Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **MOU** such that ϕ_k is a complete order isomorphism onto its image for each $k \in \mathbb{N}$. Then $\ddot{\phi}_{k,\infty}$ is a complete order isomorphism onto its image for each $k \in \mathbb{N}$.

Proof. By Remarks 3.4 and 4.4, it suffices to show that $\ddot{\phi}_{k,\infty}^{-1}$ is completely positive. Suppose $\ddot{\phi}_{k,\infty}^{(n)}(S_k) \in C_n$ for some $S_k \in M_n(\mathcal{S}_k)$. Then there exists $m > k$ such that $\phi_{k,m}^{(n)}(S_k) \in M_n(\mathcal{S}_m)^+$. Since $\phi_{k,m}$ is a complete order isomorphism onto its image, $S_k \in M_n(\mathcal{S}_k)^+$. \square

Theorem 4.6. The triple $(\ddot{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \ddot{e}_\infty)$ is an inductive limit of the inductive system $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ in **MOU**.

Proof. Suppose $(\mathcal{T}, \{\psi_k\}_{k \in \mathbb{N}})$ is a pair consisting of a matrix ordered *-vector space with matrix order unit and a family of unital completely positive maps $\psi_k : \mathcal{S}_k \rightarrow \mathcal{T}$ such that $\psi_{k+1} \circ \phi_k = \psi_k$ for all $k \in \mathbb{N}$. By Theorem 3.5, there exists a unique unital positive map $\ddot{\psi} : \ddot{\mathcal{S}}_\infty \rightarrow \mathcal{T}$ such that $\ddot{\psi} \circ \ddot{\phi}_{k,\infty} = \psi_k$ for all $k \in \mathbb{N}$. We show that $\ddot{\psi}$ is completely positive. Suppose $\ddot{\phi}_{k,\infty}^{(n)}(S_k) \in C_n$; then there exists $m > k$ such that $\phi_{k,m}^{(n)}(S_k) \in M_n(\mathcal{S}_m)^+$. Since ψ_m is completely positive,

$$\ddot{\psi}^{(n)} \circ \ddot{\phi}_{k,\infty}^{(n)}(S_k) = \psi_k^{(n)}(S_k) = \psi_m^{(n)} \circ \phi_{k,m}^{(n)}(S_k) \in M_n(\mathcal{T})^+.$$

\square

Following our general convention, we denote the triple $(\ddot{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \ddot{e}_\infty)$ by $\varinjlim_{\mathbf{MOU}} \mathcal{S}_k$.

Remark 4.7. Let $(\{\mathcal{S}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ and $(\{\mathcal{T}_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$ be inductive systems in **MOU** and let $\{\theta_k\}_{k \in \mathbb{N}}$ be a sequence of unital completely positive maps such that the following diagram commutes:

$$(16) \quad \begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 \xrightarrow{\phi_4} \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow \\ \mathcal{T}_1 & \xrightarrow{\psi_1} & \mathcal{T}_2 & \xrightarrow{\psi_2} & \mathcal{T}_3 & \xrightarrow{\psi_3} & \mathcal{T}_4 \xrightarrow{\psi_4} \dots \end{array}$$

It follows from Theorems 4.6 and 2.14 that there exists a unique unital completely positive map $\ddot{\theta} : \varinjlim_{\mathbf{MOU}} \mathcal{S}_k \rightarrow \varinjlim_{\mathbf{MOU}} \mathcal{T}_k$ such that $\ddot{\theta} \circ \ddot{\phi}_{k,\infty} = \ddot{\psi}_{k,\infty} \circ \theta_k$ for all $k \in \mathbb{N}$. In addition, it is straightforward to show that, if θ_k

is a complete order isomorphism onto its image for each $k \in \mathbb{N}$, then $\ddot{\theta}$ is a complete order isomorphism onto its image.

4.2. Inductive limits of operator systems. We now proceed to the inductive limit in the category of operator systems. Let $(\mathcal{S}_k, \{C_n^k\}_{n \in \mathbb{N}}, e_k)_{k \in \mathbb{N}}$ be a sequence of operator systems and let $\phi_k : \mathcal{S}_k \rightarrow \mathcal{S}_{k+1}$ be a unital completely positive map, $k \in \mathbb{N}$; thus,

$$(17) \quad \mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$$

is an inductive system in **OS**. Let $\mathbf{F} : \mathbf{OS} \rightarrow \mathbf{MOU}$ be the forgetful functor; consider the inductive limit $\varinjlim_{\mathbf{MOU}} \mathbf{F}(\mathcal{S}_k)$. We will show that its Archimedeanisation is an inductive limit for the inductive system (17).

Write $\varinjlim_{\mathbf{MOU}} \mathbf{F}(\mathcal{S}_k) = (\ddot{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \ddot{e}_\infty)$ (recall that $\ddot{e}_\infty = \ddot{\phi}_{k,\infty}(e_k)$, $k \in \mathbb{N}$). Let

$$N = \{s \in \ddot{\mathcal{S}}_\infty : f(s) = 0 \text{ for all } f \in S(\ddot{\mathcal{S}}_\infty)\}$$

be the *null space* of $\ddot{\mathcal{S}}_\infty$. Set

$$\mathcal{S}_\infty = \ddot{\mathcal{S}}_\infty / N,$$

write $q_S : \ddot{\mathcal{S}}_\infty \rightarrow \mathcal{S}_\infty$ for the canonical quotient map and let $\phi_{k,\infty} = q_S \circ \ddot{\phi}_{k,\infty}$. We may identify $M_n(\ddot{\mathcal{S}}_\infty / N)$ with $M_n(\ddot{\mathcal{S}}_\infty) / M_n(N)$ in a natural way. Note that, since N is closed under the involution of $\ddot{\mathcal{S}}_\infty$, the space $M_n(N)$ is closed under the involution of $M_n(\ddot{\mathcal{S}}_\infty)$.

The proof of the next lemma is analogous to that of Lemma 3.10 and is omitted.

Lemma 4.8. *Let $S_k \in M_n(\mathcal{S}_k)$. The following are equivalent:*

- (i) $\phi_{k,\infty}^{(n)}(S_k) \in (M_n(\ddot{\mathcal{S}}_\infty) / M_n(N))_h$;
- (ii) $\phi_{k,\infty}^{(n)}(S_k) = \phi_{k,\infty}^{(n)}(\text{Re}(S_k))$;
- (iii) $\phi_{k,\infty}^{(n)}(S_k) = \phi_{l,\infty}^{(n)}(S_l)$ for some $l \in \mathbb{N}$ and some $S_l \in (M_n(\mathcal{S}_l))_h$.

For each $n \in \mathbb{N}$, define

$$(18) \quad D_n = \left\{ \phi_{k,\infty}^{(n)}(S_k) \in M_n(\mathcal{S}_\infty)_h : S_k \in M_n(\mathcal{S}_k) \text{ and for each } r > 0 \text{ there exist} \right. \\ \left. l \in \mathbb{N} \text{ and } T_l \in M_n(\mathcal{S}_l) \text{ with } \ddot{\phi}_{l,\infty}^{(n)}(T_l) \in M_n(N) \right. \\ \left. \text{and } \ddot{\phi}_{k,\infty}^{(n)}(re_k^{(n)} + S_k) + \ddot{\phi}_{l,\infty}^{(n)}(T_l) \in C_n \right\}.$$

Remark 4.9. Suppose $\ddot{\phi}_{k,\infty}^{(n)}(S_k) + M_n(N) \in (M_n(\mathcal{S}_\infty))_h$. We have that $\phi_{k,\infty}^{(n)}(S_k) \in D_n$ if and only if for all $r > 0$ there exist $l \in \mathbb{N}$, $T_l \in M_n(\mathcal{S}_l)$ and $m > \max\{k, l\}$ such that $\ddot{\phi}_{l,\infty}^{(n)}(T_l) \in M_n(N)$ and $re_m^{(n)} + \phi_{k,m}^{(n)}(S_k) +$

$\phi_{l,m}^{(n)}(T_l) \in M_n(\mathcal{S}_m)^+$. We may assume without loss of generality that $l > k$, $T_l \in (M_n(\mathcal{S}_l))_h$, and $\phi_{k,m}(S_k) \in M_n(\mathcal{S}_m)_h$.

Note that the space $(\mathcal{S}_\infty, \{D_n\}_{n \in \mathbb{N}}, e_\infty)$, where $e_\infty = \phi_{k,\infty}(e_k)$ for some (and hence any) $k \in \mathbb{N}$, is the Archimedeanisation of the matrix ordered $*$ -vector space $(\check{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \check{e}_\infty)$.

Proposition 4.10. *The triple $(\mathcal{S}_\infty, \{D_n\}_{n \in \mathbb{N}}, e_\infty)$ is an operator system and $\phi_{k,\infty}$ is a unital completely positive map.*

Proof. Since $(\mathcal{S}_\infty, \{D_n\}_{n \in \mathbb{N}}, e_\infty)$ is the Archimedeanisation of the matrix ordered $*$ -vector space $(\check{\mathcal{S}}_\infty, \{C_n\}_{n \in \mathbb{N}}, \check{e}_\infty)$, it follows from [32, Proposition 3.16] that it is an operator system. By Remark 4.4, $\check{\phi}_{k,\infty}$ is a unital completely positive map. Since $q_{\mathcal{S}}$ is a unital completely positive map, we have that $\phi_{k,\infty}$ is a unital completely positive map. \square

Theorem 4.11. *The triple $(\mathcal{S}_\infty, \{D_n\}_{n \in \mathbb{N}}, e_\infty)$ is an inductive limit of the inductive system*

$$\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$$

in **OS**.

Proof. Suppose $(\mathcal{T}, \{\psi_k\}_{k \in \mathbb{N}})$ is a pair consisting of an operator system and a family of unital completely positive maps $\psi_k : \mathcal{S}_k \rightarrow \mathcal{T}$ such that $\psi_{k+1} \circ \phi_k = \psi_k$ for all $k \in \mathbb{N}$. By Theorem 4.6, there exists a unique unital completely positive map $\check{\psi} : \check{\mathcal{S}}_\infty \rightarrow \mathcal{T}$ such that $\check{\psi} \circ \check{\phi}_{k,\infty} = \psi_k$. By Theorem 2.6, there exists a unique unital completely positive map $\psi : \mathcal{S}_\infty \rightarrow \mathcal{T}$ such that $\check{\psi} = \psi \circ q_{\mathcal{S}}$. Thus

$$\psi \circ \phi_{k,\infty} = \psi \circ q_{\mathcal{S}} \circ \check{\phi}_{k,\infty} = \check{\psi} \circ \check{\phi}_{k,\infty} = \psi_k, \quad k \in \mathbb{N}.$$

\square

Using our general notational convention, we denote by $\varinjlim_{\mathbf{OS}} \mathcal{S}_k$ the inductive limit $(\mathcal{S}_\infty, \{\phi_{k,\infty}\}_{k \in \mathbb{N}})$ of the inductive system $(\{\mathcal{S}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ in the category **OS**. We often write $\mathcal{S}_\infty = \varinjlim_{\mathbf{OS}} \mathcal{S}_k$.

Remark 4.12. Let $(\{\mathcal{S}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ be an inductive system in **OS**. For each $n \in \mathbb{N}$, consider the induced inductive system

$$M_n(\mathcal{S}_1) \xrightarrow{\phi_1^{(n)}} M_n(\mathcal{S}_2) \xrightarrow{\phi_2^{(n)}} M_n(\mathcal{S}_3) \xrightarrow{\phi_3^{(n)}} M_n(\mathcal{S}_4) \xrightarrow{\phi_4^{(n)}} \dots$$

in **AOU**. Let us denote by $\phi_{k,\infty}^n$ the unital positive map associated to $\varinjlim_{\mathbf{AOU}} M_n(\mathcal{S}_k)$ so that $\phi_{k,\infty}^n : M_n(\mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{AOU}} M_n(\mathcal{S}_k)$ and $\phi_{k+1,\infty}^n \circ \phi_k^{(n)} = \phi_{k,\infty}^n$ for all $k \in \mathbb{N}$. As a consequence of Remark 2.7, one can see that $\varinjlim_{\mathbf{OS}} \mathcal{S}_k$ is the operator system with underlying $*$ -vector space $\varinjlim_{\mathbf{AOU}} \mathcal{S}_k$ such that $\phi_{k,\infty}^{(n)}(S_k) \in M_n(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)^+$ if and only if $\phi_{k,\infty}^n(S_k) \in (\varinjlim_{\mathbf{AOU}} M_n(\mathcal{S}_k))^+$.

Proposition 4.13. *Let*

$$\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$$

be an inductive system in \mathbf{OS} , and suppose that ϕ_k is a complete order embedding for each $k \in \mathbb{N}$. Then $\phi_{k,\infty}$ is a complete order embedding.

Proof. The statement follows from Proposition 3.13 and Remark 4.12. \square

Remark 4.14. Let $(\{\mathcal{S}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ and $(\{\mathcal{T}_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$ be inductive systems in \mathbf{OS} and let $\{\theta_k\}_{k \in \mathbb{N}}$ be a sequence of unital completely positive maps such that the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 \xrightarrow{\phi_4} \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow \\ \mathcal{T}_1 & \xrightarrow{\psi_1} & \mathcal{T}_2 & \xrightarrow{\psi_2} & \mathcal{T}_3 & \xrightarrow{\psi_3} & \mathcal{T}_4 \xrightarrow{\psi_4} \dots \end{array}$$

It follows from Theorems 4.6 and 2.14 that there exists a unique unital completely positive map $\theta : \varinjlim_{\mathbf{OS}} \mathcal{S}_k \rightarrow \varinjlim_{\mathbf{OS}} \mathcal{T}_k$ such that $\theta \circ \phi_{k,\infty} = \psi_{k,\infty} \circ \theta_k$ for all $k \in \mathbb{N}$. It follows from Remark 3.15 that if each θ_k is a complete order isomorphism onto its image then θ is injective.

Remark 4.15. Let $(\{\mathcal{S}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ and $(\{\mathcal{T}_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}})$ be inductive systems in \mathbf{OS} , and assume that ϕ_k and ψ_k are unital complete order embeddings, $k \in \mathbb{N}$. If $\{\theta_k\}_{k \in \mathbb{N}}$ is a sequence of unital complete order embeddings such that the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 \xrightarrow{\phi_4} \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow \\ \mathcal{T}_1 & \xrightarrow{\psi_1} & \mathcal{T}_2 & \xrightarrow{\psi_2} & \mathcal{T}_3 & \xrightarrow{\psi_3} & \mathcal{T}_4 \xrightarrow{\psi_4} \dots \end{array},$$

then $\theta : \varinjlim_{\mathbf{OS}} \mathcal{S}_k \rightarrow \varinjlim_{\mathbf{OS}} \mathcal{T}_k$ is a unital complete order embedding. The proof of the latter statement is straightforward and is therefore omitted.

Proposition 4.16. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} . For each $k \in \mathbb{N}$, let $\|\cdot\|_k$ be the norm of \mathcal{S}_k . If $s_k \in \mathcal{S}_k$ then*

$$\|\phi_{k,\infty}(s_k)\| = \lim_{l \rightarrow \infty} \|\phi_{k,l}(s_k)\|_l.$$

Proof. Let $s_k \in \mathcal{S}_k$ and suppose that $\lim_{l \rightarrow \infty} \|\phi_{k,l}(s_k)\|_l < 1$. Then there exists $m > k$ such that $\|\phi_{k,m}(s_k)\|_m < 1$. By Lemma 2.4,

$$\begin{pmatrix} e_m & \phi_{k,m}(s_k) \\ \phi_{k,m}(s_k)^* & e_m \end{pmatrix} \in M_2(\mathcal{S}_m)^+.$$

Thus,

$$\phi_{m,\infty}^{(2)} \left(\begin{pmatrix} e_m & \phi_{k,m}(s_k) \\ \phi_{k,m}(s_k)^* & e_m \end{pmatrix} \right) = \begin{pmatrix} e_\infty & \phi_{k,\infty}(s_k) \\ \phi_{k,\infty}(s_k)^* & e_\infty \end{pmatrix}$$

is an element of $M_2(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)^+$ and therefore $\|\phi_{k,\infty}(s_k)\| \leq 1$. This proves that $\|\phi_{k,\infty}(s_k)\| \leq \lim_{l \rightarrow \infty} \|\phi_{k,n}(s_k)\|_l$.

To establish the reverse inequality, suppose that $\|\phi_{k,\infty}(s_k)\| < 1$. Then

$$\begin{pmatrix} e_\infty & \phi_{k,\infty}(s_k) \\ \phi_{k,\infty}(s_k)^* & e_\infty \end{pmatrix} \in M_2(\mathcal{S}_\infty)^+.$$

Let $r > 0$. Then there exist $q \geq k$, $T_q \in M_2(\mathcal{S}_q)$ and $m > q$ such that $\ddot{\phi}_{q,\infty}^{(2)}(T_q) \in M_2(N)$ and

$$\begin{pmatrix} (1+r)e_m & \phi_{k,m}(s_k) \\ \phi_{k,m}(s_k)^* & (1+r)e_m \end{pmatrix} + \phi_{q,m}^{(2)}(T_q) \in M_2(\mathcal{S}_m)^+.$$

By Proposition 3.9 and Remark 4.12, we can choose m to have the additional property that

$$re_m^{(2)} - \phi_{q,m}^{(2)}(T_q) \in M_2(\mathcal{S}_m)^+.$$

It now follows that

$$\begin{pmatrix} (1+2r)e_p & \phi_{k,p}(s_k) \\ \phi_{k,p}(s_k)^* & (1+2r)e_p \end{pmatrix} \in M_2(\mathcal{S}_p)^+, \quad p \geq m,$$

and hence $\|\phi_{k,p}(s_k)\| \leq 1 + 2r$ for every $p \geq m$. Since r is arbitrary, we conclude that $\lim_{l \rightarrow \infty} \|\phi_{k,l}(s_k)\|_l \leq 1$. \square

4.3. Inductive limits of \mathbf{C}^* -algebras. If $(\{\mathcal{A}_k\}_{k \in \mathbb{N}}, \{\phi_k\}_{k \in \mathbb{N}})$ is an inductive system in \mathbf{C}^* then it is also an inductive system in \mathbf{OS} . In the following theorem, we compare $\varinjlim_{\mathbf{OS}} \mathcal{A}_k$ and $\varinjlim_{\mathbf{C}^*} \mathcal{A}_k$.

Theorem 4.17. *Let $\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \mathcal{A}_3 \xrightarrow{\phi_3} \mathcal{A}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{C}^* , $\mathcal{A}_0 = \varinjlim_{\mathbf{OS}} \mathcal{A}_k$ and $\mathcal{A} = \varinjlim_{\mathbf{C}^*} \mathcal{A}_k$. Then \mathcal{A}_0 is unitaly completely order isomorphic to a dense operator subsystem of \mathcal{A} .*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{A}_1 & \xrightarrow{\phi_1} & \mathcal{A}_2 & \xrightarrow{\phi_2} & \mathcal{A}_3 & \xrightarrow{\phi_3} & \mathcal{A}_4 \xrightarrow{\phi_4} \dots \\ \text{id} \downarrow & & \text{id} \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\ \mathcal{A}_1 & \xrightarrow{\phi_1} & \mathcal{A}_2 & \xrightarrow{\phi_2} & \mathcal{A}_3 & \xrightarrow{\phi_3} & \mathcal{A}_4 \xrightarrow{\phi_4} \dots \end{array}$$

By Proposition 4.16 and the definition of the inductive limit in \mathbf{C}^* , there exists an isometric linear map $\theta : \mathcal{A}_0 \rightarrow \mathcal{A}$ with dense range. It is straightforward to show that θ is completely positive.

Suppose that $\phi_{k,\infty}^{(n)}(A_k) \in M_n(\varinjlim_{\mathbf{C}^*} \mathcal{A}_k)^+$, where $A_k \in M_n(\mathcal{A}_k)$. It follows that $\phi_{k,\infty}^{(n)}(A_k) = BB^*$ where $B \in M_n(\varinjlim_{\mathbf{C}^*} \mathcal{A}_k)$. Assume $B = \lim_{p \rightarrow \infty} B^p$ where, for all $p \in \mathbb{N}$, $B^p = \phi_{m_p,\infty}^{(n)}(B_{m_p})$ for some $m_p \in \mathbb{N}$ and some $B_{m_p} \in \mathcal{A}_{m_p}$. We may assume, without loss of generality, that $A_k \in M_n(\mathcal{A}_k)_h$ and $m_p > k$ for all $p \in \mathbb{N}$. For all $r > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\|\phi_{k,\infty}^{(n)}(A_k) - \phi_{m_p,\infty}^{(n)}(B_{m_p} B_{m_p}^*)\|_{M_n(\varinjlim_{\mathbf{C}^*} \mathcal{A}_k)} < r, \quad p \geq p_0.$$

Note that

$$\begin{aligned} & \|\phi_{k,\infty}^{(n)}(A_k) - \phi_{m_p,\infty}^{(n)}(B_{m_p}B_{m_p}^*)\|_{M_n(\varinjlim_{\mathbf{C}^*} \mathcal{A}_k)} \\ &= \|\phi_{m_p,\infty}^{(n)}(\phi_{k,m_p}^{(n)}(A_k) - B_{m_p}B_{m_p}^*)\|_{M_n(\varinjlim_{\mathbf{C}^*} \mathcal{A}_k)} \\ &= \lim_{q \rightarrow \infty} \|\phi_{m_p,q}^{(n)}(\phi_{k,m_p}^{(n)}(A_k) - B_{m_p}B_{m_p}^*)\|_{M_n(\mathcal{A}_q)}. \end{aligned}$$

Fix $r > 0$ and choose $p, q \in \mathbb{N}$ such that

$$\|\phi_{m_p,q}^{(n)}(\phi_{k,m_p}^{(n)}(A_k) - B_{m_p}B_{m_p}^*)\|_{M_n(\mathcal{A}_q)} < \frac{r}{2}.$$

By [33, Corollary 5.6], the norm $\|\cdot\|_{M_n(\mathcal{A}_q)}$ agrees with the order norm on $M_n(\mathcal{A}_q)_h$; thus,

$$\frac{r}{2}e_q^{(n)} + \phi_{m_p,q}^{(n)}(\phi_{k,m_p}^{(n)}(A_k) - B_{m_p}B_{m_p}^*) \in M_n(\mathcal{A}_q)^+.$$

Since

$$\begin{aligned} \frac{r}{2}e_q^{(n)} + \phi_{m_p,q}^{(n)}(\phi_{k,m_p}^{(n)}(A_k) - B_{m_p}B_{m_p}^*) &= (re_q^{(n)} + \phi_{k,q}^{(n)}(A_k)) \\ &\quad - \left(\frac{r}{2}e_q^{(n)} + \phi_{m_p,q}^{(n)}(B_{m_p}B_{m_p}^*)\right) \end{aligned}$$

and $\frac{r}{2}e_q^{(n)} + \phi_{m_p,q}^{(n)}(B_{m_p}B_{m_p}^*) \in M_n(\mathcal{A}_q)^+$, we have that

$$re_q^{(n)} + \phi_{k,q}^{(n)}(A_k) \in M_n(\mathcal{A}_q)^+.$$

Therefore

$$r\phi_{k,\infty}^{(n)}(e_k^{(n)}) + \phi_{k,\infty}^{(n)}(A_k) = \phi_{q,\infty}^{(n)}(re_q^{(n)} + \phi_{k,q}^{(n)}(A_k)) \in M_n(\varinjlim_{\mathbf{OS}} \mathcal{A}_k)^+.$$

Since this holds for all $r > 0$, we have that $\phi_{k,\infty}^{(n)}(A_k) \in M_n(\varinjlim_{\mathbf{OS}} \mathcal{A}_k)^+$. Thus, θ is a unital complete order isomorphism onto its image. \square

Corollary 4.18. *Let $X_1 \xleftarrow{\alpha_1} X_2 \xleftarrow{\alpha_2} X_3 \xleftarrow{\alpha_3} X_4 \xleftarrow{\alpha_4} \dots$ be an inverse system in **Top** such that X_k is compact and Hausdorff, $k \in \mathbb{N}$. Let $C(X_1) \xrightarrow{\phi_1} C(X_2) \xrightarrow{\phi_2} C(X_3) \xrightarrow{\phi_3} C(X_4) \xrightarrow{\phi_4} \dots$ be the canonically induced inductive system in \mathbf{C}^* . Then there exists a unital completely order isomorphic embedding from $\varinjlim_{\mathbf{OS}} C(X_k)$ into $C(\varprojlim_{\mathbf{Top}} X_k)$.*

Proof. This follows from Proposition 4.17 and Remark 2.19. \square

4.4. Inductive limits of OMIN and OMAX. Let V_1 and V_2 be AOU spaces and $\phi : V_1 \rightarrow V_2$ be a positive map. It follows from [32, Theorem 3.4] that ϕ is a completely positive map from $\text{OMIN}(V_1)$ into $\text{OMIN}(V_2)$ and, from [32, Theorem 3.22], that ϕ is a completely positive map from $\text{OMAX}(V_1)$ into $\text{OMAX}(V_2)$. Therefore, given an inductive system

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$$

in **AOU**, we have associated inductive systems

$$\text{OMIN}(V_1) \xrightarrow{\phi_1} \text{OMIN}(V_2) \xrightarrow{\phi_2} \text{OMIN}(V_3) \xrightarrow{\phi_3} \text{OMIN}(V_4) \xrightarrow{\phi_4} \dots$$

and

$$\text{OMAX}(V_1) \xrightarrow{\phi_1} \text{OMAX}(V_2) \xrightarrow{\phi_2} \text{OMAX}(V_3) \xrightarrow{\phi_3} \text{OMAX}(V_4) \xrightarrow{\phi_4} \dots$$

in **OS**. In this section we show that the inductive limit intertwines OMAX and that it intertwines OMIN when the connecting maps are order embeddings.

Lemma 4.19. *Let V and W be AOU spaces and let $\phi : V \rightarrow W$ be a unital order embedding. Then $\phi : \text{OMIN}(V) \rightarrow \text{OMIN}(W)$ is a unital complete order embedding.*

Proof. Suppose that $\phi^{(n)}(X) \in M_n(\text{OMIN}(W))^+$ for some $X = (x_{i,j})_{i,j} \in M_n(V)$, and let $g \in S(V)$. By Lemma 3.12, there exists $\tilde{g} \in S(W)$ such that $\tilde{g} \circ \phi = g$. It follows that

$$(\langle g, x_{i,j} \rangle)_{i,j} = (\langle \tilde{g}, \phi(x_{i,j}) \rangle)_{i,j} \in M_n^+.$$

By Theorem 2.10, $X \in M_n(\text{OMIN}(V))^+$. \square

Theorem 4.20. *Let $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **AOU** such that each ϕ_k is a unital order embedding. Then $\text{OMIN}(\varinjlim_{\text{AOU}} V_k)$ is unittally completely order isomorphic to $\varinjlim_{\text{OS}} \text{OMIN}(V_k)$.*

Proof. Let

$$S(V_1) \xleftarrow{\phi'_1} S(V_2) \xleftarrow{\phi'_2} S(V_3) \xleftarrow{\phi'_3} S(V_4) \xleftarrow{\phi'_4} \dots$$

be the corresponding inverse system in **Top**. Note that each ϕ'_k is surjective. By Proposition 3.16, there exists a homeomorphism $\alpha : S(\varinjlim_{\text{AOU}} V_k) \rightarrow \varprojlim_{\text{Top}} S(V_k)$. Let $\hat{\alpha} : C(\varprojlim_{\text{Top}} S(V_k)) \rightarrow C(S(\varinjlim_{\text{AOU}} V_k))$ be the unital $*$ -isomorphism induced by α .

Consider, in addition, the induced inductive system in **C*** with $*$ -isomorphic embeddings

$$C(S(V_1)) \xrightarrow{\alpha_1} C(S(V_2)) \xrightarrow{\alpha_2} C(S(V_3)) \xrightarrow{\alpha_3} C(S(V_4)) \xrightarrow{\alpha_4} \dots$$

By Corollary 4.18, there exists a unital complete order embedding

$$\beta : \varinjlim_{\text{OS}} C(S(V_k)) \rightarrow C(\varprojlim_{\text{Top}} S(V_k)).$$

By Theorem 2.10, for each $k \in \mathbb{N}$ the natural inclusion $\iota_k : \text{OMIN}(V_k) \rightarrow C(S(V_k))$ is a unital completely order isomorphic embedding.

The diagram

$$\begin{array}{ccccccc} \text{OMIN}(V_1) & \xrightarrow{\phi_1} & \text{OMIN}(V_2) & \xrightarrow{\phi_2} & \text{OMIN}(V_3) & \xrightarrow{\phi_3} & \dots \\ \iota_1 \downarrow & & \iota_2 \downarrow & & \iota_3 \downarrow & & \\ C(S(V_1)) & \xrightarrow{\alpha_1} & C(S(V_2)) & \xrightarrow{\alpha_2} & C(S(V_3)) & \xrightarrow{\alpha_3} & \dots \end{array}$$

commutes since $\alpha_k|_{\text{OMIN}(V_k)} = \phi_k$. By Remark 4.15, there exists a unital complete order embedding

$$\iota : \varinjlim_{\text{OS}} \text{OMIN}(V_k) \rightarrow \varinjlim_{\text{OS}} C(S(V_k)).$$

Therefore

$$\hat{\alpha} \circ \beta \circ \iota : \varinjlim_{\mathbf{OS}} \text{OMIN}(V_k) \rightarrow C(S(\varinjlim_{\mathbf{AOU}} V_k))$$

is a unital completely order isomorphic embedding. Thus, $\varinjlim_{\mathbf{OS}} \text{OMIN}(V_k)$ is completely order isomorphic to an operator subsystem \mathcal{T} of the C^* -algebra $C(S(\varinjlim_{\mathbf{AOU}} V_k))$. By Theorem 2.10, \mathcal{T} is completely order isomorphic to $\text{OMIN}(\varinjlim_{\mathbf{AOU}} V_k)$. \square

Denote by OMAX the functor from \mathbf{AOU} to \mathbf{OS} , sending V to $\text{OMAX}(V)$. As pointed out in Section 2.3, OMAX is a left adjoint to the forgetful functor $\mathbf{F} : \mathbf{OS} \rightarrow \mathbf{AOU}$. The well-known fact that left adjoints commute with colimits [25] has the following immediate consequence, which complements Theorem 4.20.

Theorem 4.21. *Let $V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{AOU} . Then $\text{OMAX}(\varinjlim_{\mathbf{AOU}} V_k)$ is unitaly completely order isomorphic to $\varinjlim_{\mathbf{OS}} \text{OMAX}(V_k)$.*

4.5. Inductive limits of universal C^* -algebras. In this section, we consider the universal C^* -algebra of an inductive limit operator system; we show in Theorem 4.23 that C_u^* commutes with $\varinjlim_{\mathbf{OS}}$ when the connecting maps are complete order embeddings. The result is well-known in the case of closed operator systems (see [24, Proposition 2.4]). We have decided to include complete arguments in order to keep the exposition self-contained.

The following lemma was established in [22].

Lemma 4.22 ([22]). *Let \mathcal{S} and \mathcal{T} be operator systems with universal C^* -algebras $(C_u^*(\mathcal{S}), \iota_{\mathcal{S}})$ and $(C_u^*(\mathcal{T}), \iota_{\mathcal{T}})$, respectively, and let $\phi : \mathcal{S} \rightarrow \mathcal{T}$ be a unital complete order embedding. Then the $*$ -homomorphism $\tilde{\phi} : C_u^*(\mathcal{S}) \rightarrow C_u^*(\mathcal{T})$ with the property that $\tilde{\phi} \circ \iota_{\mathcal{S}} = \iota_{\mathcal{T}} \circ \phi$ is injective.*

Clearly, if $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ is an inductive system in \mathbf{OS} then

$$C_u^*(\mathcal{S}_1) \xrightarrow{\tilde{\phi}_1} C_u^*(\mathcal{S}_2) \xrightarrow{\tilde{\phi}_2} C_u^*(\mathcal{S}_3) \xrightarrow{\tilde{\phi}_3} C_u^*(\mathcal{S}_4) \xrightarrow{\tilde{\phi}_4} \dots$$

is an inductive system in \mathbf{C}^* . Let $\pi_k : C_u^*(\mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)$ be the canonical unital $*$ -homomorphism, $k \in \mathbb{N}$.

Theorem 4.23. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} such that each ϕ_k is a unital complete order embedding. Then $C_u^*(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)$ is $*$ -isomorphic to $\varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)$.*

Proof. Set $\mathcal{S}_\infty = \varinjlim_{\mathbf{OS}} \mathcal{S}_k$ and let $\iota_{\mathcal{S}_\infty} : \mathcal{S}_\infty \rightarrow C_u^*(\mathcal{S}_\infty)$ be the canonical embedding. Consider the following commutative diagram

$$(19) \quad \begin{array}{ccccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 & \xrightarrow{\phi_4} & \dots \\ \iota_1 \downarrow & & \iota_2 \downarrow & & \iota_3 \downarrow & & \iota_4 \downarrow & & \\ C_u^*(\mathcal{S}_1) & \xrightarrow{\tilde{\phi}_1} & C_u^*(\mathcal{S}_2) & \xrightarrow{\tilde{\phi}_2} & C_u^*(\mathcal{S}_3) & \xrightarrow{\tilde{\phi}_3} & C_u^*(\mathcal{S}_4) & \xrightarrow{\tilde{\phi}_4} & \dots \end{array}$$

By Lemma 4.22, all maps in (19) are unital complete order embeddings. By Remark 4.15, there exists a unique unital complete order embedding $\iota : \mathcal{S}_\infty \rightarrow \varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k)$ such that

$$(20) \quad \iota \circ \phi_{k,\infty} = \pi_k \circ \iota_k, \quad k \in \mathbb{N}.$$

By Proposition 4.17, the natural map $\text{id} : \varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)$ is a unital complete order embedding; thus, $\iota : \mathcal{S}_\infty \rightarrow \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)$ is a unital complete order embedding.

By Proposition 2.11, there exists a unique unital *-homomorphism

$$\nu : C_u^*(\mathcal{S}_\infty) \rightarrow \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)$$

such that

$$(21) \quad \nu \circ \iota_{\mathcal{S}_\infty} = \iota.$$

Note that $\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty} : \mathcal{S}_k \rightarrow C_u^*(\mathcal{S}_\infty)$ is a unital completely order isomorphic embedding, $k \in \mathbb{N}$. By Proposition 2.11, there exists a unital *-homomorphism

$$\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}} : C_u^*(\mathcal{S}_k) \rightarrow C_u^*(\mathcal{S}_\infty)$$

such that

$$(22) \quad (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}}) \circ \iota_k = \iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}, \quad k \in \mathbb{N}.$$

By (22),

$$\begin{aligned} (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k+1,\infty}}) \circ \tilde{\phi}_k \circ \iota_k &= (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k+1,\infty}}) \circ \iota_{k+1} \circ \phi_k \\ &= \iota_{\mathcal{S}_\infty} \circ \phi_{k+1,\infty} \circ \phi_k = \iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty} = (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}}) \circ \iota_k \end{aligned}$$

for all $k \in \mathbb{N}$. By the universal property of the inductive limit in the category of C*-algebras, there exists a unique unital *-homomorphism $\mu : \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k) \rightarrow C_u^*(\mathcal{S}_\infty)$ such that

$$(23) \quad \mu \circ \pi_k = (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}}), \quad k \in \mathbb{N}.$$

Note that $\mu \circ \nu = \text{id}_{C_u^*(\mathcal{S}_\infty)}$ and $\nu \circ \mu = \text{id}_{\varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k)}$. Indeed, by (20), (21), (22) and (23),

$$\begin{aligned} \mu \circ \nu \circ \iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty} &= \mu \circ \iota \circ \phi_{k,\infty} = \mu \circ \pi_k \circ \iota_k = (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}}) \circ \iota_k \\ &= \iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty} \end{aligned}$$

and

$$\begin{aligned} \nu \circ \mu \circ \pi_k \circ \iota_k &= \nu \circ (\widetilde{\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}}) \circ \iota_k = \nu \circ \iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty} = \iota \circ \phi_{k,\infty} \\ &= \pi_k \circ \iota_k. \end{aligned}$$

Since $\mu \circ \nu$ and $\nu \circ \mu$ coincide with the identities on dense operator systems, generating the corresponding C^* -algebras, we have that μ is a $*$ -isomorphism. \square

Theorems 4.17 and 4.23 have the following straightforward corollary.

Corollary 4.24. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} such that each ϕ_k is a unital completely order isomorphic embedding. Then $\varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k)$ is unital completely order isomorphic to an operator subsystem of $C_u^*(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)$.*

4.6. Quotients of inductive limits of operator systems. In this subsection, we relate inductive limits with the quotient theory of operator systems. We first recall the basic facts about quotient operator systems, as developed in [19].

Let \mathcal{S} be an operator system and let $J \subseteq \mathcal{S}$ be a subspace. If there exists an operator system \mathcal{T} and a unital completely positive map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ such that $J = \ker \phi$, then we say that J is a *kernel*. If J is a kernel, we let $q : \mathcal{S} \rightarrow \mathcal{S}/J$ be the quotient map and equip the quotient vector space \mathcal{S}/J with the involution given by $(x + J)^* = x^* + J$. For $n \in \mathbb{N}$, let

$$\begin{aligned} C_n(\mathcal{S}/J) &= \left\{ (x_{i,j} + J) \in M_n(\mathcal{S}/J) : \forall r > 0 \exists k_{i,j} \in J \right. \\ &\quad \left. \text{such that } re^{(n)} + (x_{i,j} + k_{i,j})_{i,j} \in M_n(\mathcal{S})^+ \right\}. \end{aligned}$$

It was shown in [19, Section 3] that $(\mathcal{S}/J, \{C_n(\mathcal{S}/J)\}_{n \in \mathbb{N}}, e + J)$ is an operator system (called henceforth a *quotient operator system*); moreover, the following holds:

Theorem 4.25. *Let \mathcal{S} and \mathcal{T} be operator systems and let J be a kernel in \mathcal{S} . If $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map with $J \subseteq \ker \phi$ then the map $\tilde{\phi} : \mathcal{S}/J \rightarrow \mathcal{T}$, defined by the identity $\tilde{\phi} \circ q = \phi$, is unital and completely positive.*

Furthermore, if \mathcal{P} is an operator system and $\psi : \mathcal{S} \rightarrow \mathcal{P}$ is a unital completely positive map such that whenever \mathcal{T} is an operator system and $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map with $J \subseteq \ker \phi$ there exists a unique unital completely positive map $\tilde{\phi} : \mathcal{P} \rightarrow \mathcal{T}$ with the property that $\tilde{\phi} \circ \psi = \phi$, then there exists a complete order isomorphism $\varphi : \mathcal{P} \rightarrow \mathcal{S}/J$ such that $\varphi \circ \psi = q$.

If \mathcal{X} is a (not necessarily complete) operator space and Y is a closed subspace of \mathcal{X} , then the quotient \mathcal{X}/Y has a canonical operator space structure given by assigning $M_n(\mathcal{X}/Y)$ the norm arising from the identification

$M_n(\mathcal{X}/Y) = M_n(\mathcal{X})/M_n(Y)$, that is, by setting

$$(24) \quad \|(x_{i,j} + Y)\|_{\text{osp}} = \inf \{ \|x_{i,j} + y_{i,j}\|_{M_n(\mathcal{X})} : y_{i,j} \in Y \}, \quad (x_{i,j}) \in M_n(\mathcal{X}).$$

If \mathcal{S} is an operator system and J is a kernel, then \mathcal{S}/J can be equipped, on one hand, with the operator space structure inherited from the quotient operator system \mathcal{S}/J (whose norms are denoted by $\|\cdot\|$ or $\|\cdot\|_{\mathcal{S}/J}$). and, on the other hand, with the operator space structure given by (24). In general, $\|x\| \leq \|x\|_{\text{osp}}$ for every $x \in M_n(\mathcal{S}/J)$ and every $n \in \mathbb{N}$. If there exists a constant $C > 0$ such that $\|x\|_{\text{osp}} \leq C\|x\|$ for every $x \in \mathcal{S}/J$, we will call J a *C-uniform kernel*. We note that a characterisation of a subclass of *C-uniform kernels* (namely, of the kernels J for which the operator system and the operator space quotient by J are completely boundedly isomorphic) in terms of decomposability conditions was given in [19, Theorem 4.8].

Suppose that $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \cdots$ is an inductive system in **OS** and that, for each $k \in \mathbb{N}$, J_k is a kernel in \mathcal{S}_k such that $\phi_k(J_k) \subseteq J_{k+1}$. Let $q_k : \mathcal{S}_k \rightarrow \mathcal{S}_k/J_k$ be the quotient map. By Theorem 4.25, there is a natural inductive system in **OS**,

$$(25) \quad \mathcal{S}_1/J_1 \xrightarrow{\psi_1} \mathcal{S}_2/J_2 \xrightarrow{\psi_2} \mathcal{S}_3/J_3 \xrightarrow{\psi_3} \mathcal{S}_4/J_4 \xrightarrow{\psi_4} \cdots,$$

such that

$$(26) \quad \psi_k \circ q_k = q_{k+1} \circ \phi_k, \quad k \in \mathbb{N}.$$

In this subsection we prove that if each of the J_k is *C-uniform*, then the inductive limit of (25) is a quotient operator system.

Lemma 4.26. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \cdots$ be an inductive system in **OS**. Let $C > 0$ and, for each $k \in \mathbb{N}$, let J_k be a *C-uniform kernel* in \mathcal{S}_k such that $\phi_k(J_k) \subseteq J_{k+1}$. Then $\varinjlim_{\text{OS}} J_k \stackrel{\text{def}}{=} \overline{\cup_{k \in \mathbb{N}} \phi_{k,\infty}(J_k)}$ is a kernel in $\varinjlim_{\text{OS}} \mathcal{S}_k$.*

Proof. Set $\mathcal{S}_\infty = \varinjlim_{\text{OS}} \mathcal{S}_k$ and $J = \varinjlim J_k$; clearly, J is a closed subspace of \mathcal{S}_∞ . Note that $q_{k+1} \circ \phi_k : \mathcal{S}_k \rightarrow \mathcal{S}_{k+1}/J_{k+1}$ is a unital completely positive map. Consider the commuting diagram

$$\begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 \xrightarrow{\phi_4} \cdots \\ q_1 \downarrow & & q_2 \downarrow & & q_3 \downarrow & & q_4 \downarrow \\ \mathcal{S}_1/J_1 & \xrightarrow{\psi_1} & \mathcal{S}_2/J_2 & \xrightarrow{\psi_2} & \mathcal{S}_3/J_3 & \xrightarrow{\psi_3} & \mathcal{S}_4/J_4 \xrightarrow{\psi_4} \cdots \end{array}$$

By Theorem 4.11, there exists a (unique) unital completely positive map $q : \mathcal{S}_\infty \rightarrow \varinjlim_{\text{OS}} (\mathcal{S}_k/J_k)$, such that

$$(27) \quad q \circ \phi_{k,\infty} = \psi_{k,\infty} \circ q_k, \quad k \in \mathbb{N}.$$

We show that $\ker q = J$. Since $\ker q$ is closed, in order to prove that $J \subseteq \ker q$, it suffices to show that $\cup_{k \in \mathbb{N}} \phi_{k,\infty}(J_k) \subseteq \ker q$. But, if $y_k \in J_k$ then

$q \circ \phi_{k,\infty}(y_k) = \psi_{k,\infty} \circ q_k(y_k) = 0$. Now suppose that $\phi_{k,\infty}(s_k) \in \ker q$ for some $s_k \in \mathcal{S}_k$; then $\psi_{k,\infty} \circ q_k(s_k) = q \circ \phi_{k,\infty}(s_k) = 0$. By Proposition 4.16,

$$\lim_{m \rightarrow \infty} \|q_m \circ \phi_{k,m}(s_k)\|_{\mathcal{S}_m/J_m} = \lim_{m \rightarrow \infty} \|\psi_{k,m} \circ q_k(s_k)\|_{\mathcal{S}_m/J_m} = 0.$$

For $l \in \mathbb{N}$, let $m_l \in \mathbb{N}$ be such that

$$\|q_m \circ \phi_{k,m}(s_k)\|_{\mathcal{S}_m/J_m} < \frac{1}{l}, \quad m \geq m_l.$$

Since J_{m_l} is C -uniform, there exists $y_{m_l} \in J_{m_l}$ such that

$$\|\phi_{k,m_l}(s_k) + y_{m_l}\|_{\mathcal{S}_{m_l}} < \frac{C}{l}.$$

The map $\phi_{m_l,\infty}$ is unital and completely positive; therefore it is contractive and hence, for all $l \in \mathbb{N}$,

$$\|\phi_{m_l,\infty}(\phi_{k,m_l}(s_k) + y_{m_l})\|_{\mathcal{S}_\infty} \leq \|\phi_{k,m_l}(s_k) + y_{m_l}\|_{\mathcal{S}_{m_l}} < \frac{C}{l}.$$

Thus, $\phi_{k,\infty}(s_k) = -\lim_{l \rightarrow \infty} \phi_{m_l,\infty}(y_{m_l})$; on the other hand, $\phi_{m_l,\infty}(y_{m_l}) \in J$ for every l , and therefore $\ker q \subseteq J$. \square

In view of Lemma 4.26, the operator system $(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)/(\varinjlim J_k)$ is well-defined. We let $\gamma : \varinjlim_{\mathbf{OS}} \mathcal{S}_k \rightarrow (\varinjlim_{\mathbf{OS}} \mathcal{S}_k)/(\varinjlim J_k)$ be the corresponding quotient map.

Theorem 4.27. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} . Let $C > 0$ and J_k be a C -uniform kernel in \mathcal{S}_k such that $\phi_k(J_k) \subseteq J_{k+1}$, $k \in \mathbb{N}$. Then there exists a unital complete order isomorphism $\rho : \varinjlim_{\mathbf{OS}} (\mathcal{S}_k/J_k) \rightarrow (\varinjlim_{\mathbf{OS}} \mathcal{S}_k)/(\varinjlim J_k)$ such that*

$$\rho \circ \psi_{k,\infty} \circ q_k = \gamma \circ \phi_{k,\infty}, \quad k \in \mathbb{N}.$$

Proof. Set $\mathcal{S}_\infty = \varinjlim_{\mathbf{OS}} \mathcal{S}_k$. Let \mathcal{T} be an operator system and $\theta : \mathcal{S}_\infty \rightarrow \mathcal{T}$ be a unital completely positive map such that $\varinjlim J_k \subseteq \ker \theta$; then $\theta \circ \phi_{k,\infty} : \mathcal{S}_k \rightarrow \mathcal{T}$ is a unital completely positive map, $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ and suppose $y_k \in J_k$; by definition, $\phi_{k,\infty}(y_k) \in \varinjlim J_k$ and so $\theta \circ \phi_{k,\infty}(y_k) = 0$. Thus, $J_k \subseteq \ker(\theta \circ \phi_{k,\infty})$. By Theorem 4.25, there exists a unique unital completely positive map $(\theta \circ \phi_{k,\infty}) : \mathcal{S}_k/J_k \rightarrow \mathcal{T}$ such that

$$(28) \quad (\theta \circ \phi_{k,\infty}) \circ q_k = \theta \circ \phi_{k,\infty}, \quad k \in \mathbb{N}.$$

By (26) and (28),

$$\begin{aligned} (\theta \circ \phi_{k+1,\infty}) \circ \psi_k \circ q_k &= (\theta \circ \phi_{k+1,\infty}) \circ q_{k+1} \circ \phi_k = \theta \circ \phi_{k+1,\infty} \circ \phi_k \\ &= \theta \circ \phi_{k,\infty} = (\theta \circ \phi_{k,\infty}) \circ q_k \end{aligned}$$

for every $k \in \mathbb{N}$. By Theorem 4.11, there exists a unique unital completely positive map $\tilde{\theta} : \varinjlim_{\mathbf{OS}} (\mathcal{S}_k/J_k) \rightarrow \mathcal{T}$ such that

$$(29) \quad \tilde{\theta} \circ \psi_{k,\infty} = (\theta \circ \phi_{k,\infty}), \quad k \in \mathbb{N}.$$

By (27), (28) and (29),

$$\tilde{\theta} \circ q \circ \phi_{k,\infty} = \tilde{\theta} \circ \psi_{k,\infty} \circ q_k = (\widetilde{\theta \circ \phi_{k,\infty}}) \circ q_k = \theta \circ \phi_{k,\infty}, \quad k \in \mathbb{N},$$

where $q : \mathcal{S}_\infty \rightarrow \varinjlim \mathbf{OS}(\mathcal{S}_k/J_k)$ is the map defined through (27). Thus, $\tilde{\theta} \circ q = \theta$. By Theorem 4.25, there exists a unital complete order isomorphism $\rho : \varinjlim \mathbf{OS}(\mathcal{S}_k/J_k) \rightarrow (\varinjlim \mathbf{OS} \mathcal{S}_k)/(\varinjlim J_k)$ such that $\rho \circ q = \gamma$. This implies that $\rho \circ q \circ \phi_{k,\infty} = \gamma \circ \phi_{k,\infty}$ which, by virtue of (27), means that $\rho \circ \psi_{k,\infty} \circ q_k = \gamma \circ \phi_{k,\infty}$, $k \in \mathbb{N}$. \square

Remark We do not know if Theorem 4.27 holds true without the assumption that the kernels J_k be C -uniform. In fact, the problem lies in determining if, in this case, the subspace $\varinjlim J_k$ is still a kernel in $\varinjlim \mathbf{OS} \mathcal{S}_k$.

4.7. Inductive limits and tensor products. Let

$$(30) \quad \mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$$

be an inductive system in \mathbf{OS} . Let \mathcal{T} be an operator system; for any functorial operator system tensor product μ , we may define the following inductive system in \mathbf{OS} :

$$(31) \quad \mathcal{S}_1 \otimes_\mu \mathcal{T} \xrightarrow{\phi_1 \otimes \text{id}_\mathcal{T}} \mathcal{S}_2 \otimes_\mu \mathcal{T} \xrightarrow{\phi_2 \otimes \text{id}_\mathcal{T}} \mathcal{S}_3 \otimes_\mu \mathcal{T} \xrightarrow{\phi_3 \otimes \text{id}_\mathcal{T}} \mathcal{S}_4 \otimes_\mu \mathcal{T} \xrightarrow{\phi_4 \otimes \text{id}_\mathcal{T}} \dots$$

We are interested to know if $\varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$ is completely order isomorphic to $(\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_\mu \mathcal{T}$. We first discuss the canonical linear isomorphism between these vector spaces.

Recalling the notation from Subsection 3.3, let N be the null space for the inductive system (30) and let N_μ be the null space for the inductive system (31). Let $\psi_k = \phi_k \otimes \text{id}_\mathcal{T}$ and $\psi_{k,\infty} : \mathcal{S}_k \otimes_\mu \mathcal{T} \rightarrow \varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$ be the unital completely positive map associated to the inductive system (31).

Lemma 4.28. *If $x \in (\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T}$ then there exist $k, n \in \mathbb{N}$, $s_k^i \in \mathcal{S}_k$ and $t^i \in \mathcal{T}$, $1 \leq i \leq n$, such that the set $\{t^i\}_{i=1}^n$ is linearly independent and*

$$x = \sum_{i=1}^n \phi_{k,\infty}(s_k^i) \otimes t^i.$$

Proof. Since $\varinjlim \mathbf{OS} \mathcal{S}_k = \cup_{k \in \mathbb{N}} \phi_{k,\infty}(\mathcal{S}_k)$, there exists $n \in \mathbb{N}$, $k_i \in \mathbb{N}$, $s_{k_i} \in \mathcal{S}_{k_i}$ and $t^i \in \mathcal{T}$, $i = 1, \dots, n$, such that $x = \sum_{i=1}^n \phi_{k_i,\infty}(s_{k_i}) \otimes t^i$. Let $k = \max\{k_i : 1 \leq i \leq n\}$ and $s_k^i = \phi_{k_i,k}(s_{k_i})$, $i = 1, \dots, n$. Choosing n to be minimal with this property ensures that $\{t^i\}_{i=1}^n$ is linearly independent. \square

Proposition 4.29. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} . Let \mathcal{T} be an operator system and μ be a functorial operator system tensor product. Then the mapping $\tilde{\alpha} : (\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T} \rightarrow \varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$ given by*

$$(32) \quad \tilde{\alpha} \circ (\phi_{k,\infty} \otimes \text{id}_\mathcal{T}) = \psi_{k,\infty}, \quad k \in \mathbb{N},$$

is a well-defined linear bijection.

Proof. Suppose that $(\phi_{k,\infty}(s_k), t_1) = (\phi_{l,\infty}(s_l), t_2)$ for some $s_k \in \mathcal{S}_k, s_l \in \mathcal{S}_l$ where $k < l$ and $t_1, t_2 \in \mathcal{T}$. Then $\phi_{l,\infty}(\phi_{k,l}(s_k) - s_l) = 0$ and $t_1 = t_2$. By Proposition 4.16, $\lim_{p \rightarrow \infty} \|\phi_{l,p}(\phi_{k,l}(s_k) - s_l)\|_{\mathcal{S}_p} = 0$ and thus

$$\begin{aligned} & \lim_{p \rightarrow \infty} \|\psi_{l,p}(\psi_{k,l}((s_k \otimes t_1) - s_l \otimes t_2))\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} \\ &= \lim_{p \rightarrow \infty} \|\psi_{l,p}((\phi_{k,l}(s_k) - s_l) \otimes t_1)\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} \\ &= \lim_{p \rightarrow \infty} \|\phi_{l,p}(\phi_{k,l}(s_k) - s_l) \otimes t_1\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} \\ &\leq \|t_1\|_{\mathcal{T}} \lim_{p \rightarrow \infty} \|\phi_{l,p}(\phi_{k,l}(s_k) - s_l)\|_{\mathcal{S}_p} = 0, \end{aligned}$$

where the last inequality follows from [20, Proposition 3.4]. By Proposition 4.16, $\psi_{l,\infty}(\psi_{k,l}(s_k \otimes t_1) - s_l \otimes t_2) = 0$ and hence $\psi_{k,\infty}(s_k \otimes t_1) = \psi_{l,\infty}(s_l \otimes t_2)$. It follows that the map $\alpha : (\varinjlim \mathbf{OS} \mathcal{S}_k) \times \mathcal{T} \rightarrow \varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$, given by $\alpha(\phi_{k,\infty}(s_k), t) = \psi_{k,\infty}(s_k \otimes t)$, is well-defined. The map α is clearly bilinear, and its linearisation $\tilde{\alpha} : (\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T} \rightarrow \varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$ satisfies

$$\tilde{\alpha}(\phi_{k,\infty}(s_k) \otimes t) = \psi_{k,\infty}(s_k \otimes t), \quad s_k \in \mathcal{S}_k, t \in \mathcal{T}, k \in \mathbb{N}.$$

We show that $\tilde{\alpha}$ is bijective. To show that $\tilde{\alpha}$ is surjective, suppose that $y \in \varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_\mu \mathcal{T})$ and write

$$y = \psi_{k,\infty}\left(\sum_{i=1}^n s_k^i \otimes t^i\right),$$

where $s_k^i \in \mathcal{S}_k, k \in \mathbb{N}$, and $t^i \in \mathcal{T}, 1 \leq i \leq n$. Then

$$\sum_{i=1}^n \phi_{k,\infty}(s_k^i) \otimes t^i \in (\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T}$$

and

$$\begin{aligned} \tilde{\alpha}\left(\sum_{i=1}^n \phi_{k,\infty}(s_k^i) \otimes t^i\right) &= \tilde{\alpha} \circ (\phi_{k,\infty} \otimes \text{id})\left(\sum_{i=1}^n s_k^i \otimes t^i\right) \\ &= \sum_{i=1}^n \tilde{\alpha}(\phi_{k,\infty}(s_k^i) \otimes t^i) = \sum_{i=1}^n \psi_{k,\infty}(s_k^i \otimes t^i) \\ &= \psi_{k,\infty}\left(\sum_{i=1}^n s_k^i \otimes t^i\right) = y. \end{aligned}$$

To see that $\tilde{\alpha}$ is injective, let $x \in (\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T}$ with $\tilde{\alpha}(x) = 0$. Using Lemma 4.28, write $x = \sum_{i=1}^n \phi_{k,\infty}(s_k^i) \otimes t^i$ for some $k \in \mathbb{N}, s_k^i \in \mathcal{S}_k, 1 \leq i \leq n$, and a linearly independent family $\{t^i\}_{i=1}^n \subseteq \mathcal{T}$. Since

$$\tilde{\alpha}\left(\sum_{i=1}^n \phi_{k,\infty}(s_k^i) \otimes t^i\right) = \psi_{k,\infty}\left(\sum_{i=1}^n s_k^i \otimes t^i\right),$$

it follows by Proposition 4.16 that

$$\lim_{p \rightarrow \infty} \left\| \sum_{i=1}^n \phi_{k,p}(s_k^i) \otimes t^i \right\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} = \lim_{p \rightarrow \infty} \left\| \psi_{k,p} \left(\sum_{i=1}^n s_k^i \otimes t^i \right) \right\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} = 0.$$

Let $W = \text{span}\{t^1, t^2, \dots, t^n\} \subseteq \mathcal{T}$ and define, for each $l = 1, \dots, n$, a linear functional $f_l : W \rightarrow \mathbb{C}$ by letting

$$f_l(t^i) = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{if } i \neq l. \end{cases}$$

Each f_l is bounded and may be extended to a bounded functional $\tilde{f}_l : \mathcal{T} \rightarrow \mathbb{C}$. It follows from [20, Proposition 3.7] that for any $k \in \mathbb{N}$ and $1 \leq l \leq n$, $\|\text{id}_{\mathcal{S}_k} \otimes \tilde{f}_l\| \leq \|\tilde{f}_l\|$. Therefore, for each $l = 1, \dots, n$,

$$\begin{aligned} \lim_{p \rightarrow \infty} \|\phi_{k,p}(s_k^l)\|_{\mathcal{S}_p} &= \lim_{p \rightarrow \infty} \left\| (\text{id}_{\mathcal{S}_k} \otimes \tilde{f}_l) \left(\sum_{i=1}^n \phi_{k,p}(s_k^i) \otimes t^i \right) \right\|_{\mathcal{S}_p} \\ &\leq \|\tilde{f}_l\| \lim_{p \rightarrow \infty} \left\| \sum_{i=1}^n \phi_{k,p}(s_k^i) \otimes t^i \right\|_{\mathcal{S}_p \otimes_\mu \mathcal{T}} = 0. \end{aligned}$$

By Proposition 4.16, $\phi_{k,\infty}(s_k^l) = 0$ for each $l = 1, \dots, n$ and hence $x = 0$. \square

Throughout this section, unless otherwise specified, we let $\tilde{\alpha}$ denote the map defined by (32).

Remark 4.30. Let $k \in \mathbb{N}$ and $R \in M_n(\mathcal{S}_k \otimes_\mu \mathcal{T})$. We have that $\ddot{\psi}_{k,\infty}^{(n)}(R) \in M_n(N_\mu)$ if and only if $(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(R) = 0$.

Proof. If $R = (r_{i,j})_{i,j} \in M_n(\mathcal{S}_k \otimes_\mu \mathcal{T})$ and $\ddot{\psi}_{k,\infty}^{(n)}(R) \in M_n(N_\mu)$ then $\psi_{k,\infty}(r_{i,j}) = 0$ for all i, j and hence, by the injectivity of the map $\tilde{\alpha}$, established in Proposition 4.29, we have that $(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})(r_{i,j}) = 0$ for all i, j . Thus, $(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(R) = 0$. Conversely, if $(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(R) = 0$ then $\psi_{k,\infty}(r_{i,j}) = \tilde{\alpha}((\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})(r_{i,j})) = 0$ for all i, j and hence $\ddot{\psi}_{k,\infty}^{(n)}(R) \in M_n(N_\mu)$. \square

Theorem 4.31. Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **OS**. Let \mathcal{T} be an operator system and μ be a functorial operator system tensor product. Then the inverse $\tilde{\alpha}^{-1} : \varinjlim_{\text{OS}} (\mathcal{S}_k \otimes_\mu \mathcal{T}) \rightarrow (\varinjlim_{\text{OS}} \mathcal{S}_k) \otimes_\mu \mathcal{T}$ of the map $\tilde{\alpha}$ is a unital completely positive map.

Proof. Suppose $\psi_{k,\infty}^{(n)}(R) \in M_n(\varinjlim_{\text{OS}} (\mathcal{S}_k \otimes \mathcal{T}))^+$ for some $R \in M_n(\mathcal{S}_k \otimes_\mu \mathcal{T})_h$, $k \in \mathbb{N}$. Then for every $r > 0$ there exist $l \in \mathbb{N}$, $P \in \mathcal{S}_l \otimes_\mu \mathcal{T}$ and $m > \max\{k, l\}$ such that $\ddot{\psi}_{l,\infty}^{(n)}(P) \in M_n(N_\mu)$ and

$$r(e_m \otimes e_{\mathcal{T}})^{(n)} + \psi_{k,m}^{(n)}(R) + \psi_{l,m}^{(n)}(P) \in M_n(\mathcal{S}_m \otimes_\mu \mathcal{T})^+.$$

By Remark 4.30,

$$\begin{aligned} & r(\phi_{k,\infty}(e_k) \otimes e_{\mathcal{T}})^{(n)} + (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(R) \\ &= (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(r(e_m \otimes e_{\mathcal{T}})^{(n)} + \psi_{k,m}^{(n)}(R) + \psi_{l,m}^{(n)}(P)) \\ &\in M_n((\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T})^+. \end{aligned}$$

Since this holds for all $r > 0$, it follows that

$$(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(R) \in M_n((\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T})^+.$$

Since $\tilde{\alpha}^{-1} \circ \psi_{k,\infty} = \phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}$, the proof is complete. \square

Theorem 4.32. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} such that each ϕ_k is a complete order isomorphism onto its image. Let \mathcal{T} be an operator system and μ be a functorial, injective operator system tensor product. Then the map $\tilde{\alpha} : (\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T} \rightarrow \varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_{\mu} \mathcal{T})$ is a unital complete order isomorphism.*

Proof. Note that the maps $\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}$, $k \in \mathbb{N}$, are completely positive and

$$(\phi_{k+1,\infty} \otimes \text{id}_{\mathcal{T}}) \circ (\phi_k \otimes \text{id}_{\mathcal{T}}) = \phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}, \quad k \in \mathbb{N}.$$

We will show that the pair

$$\left((\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T}, \{\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}\}_{k \in \mathbb{N}} \right)$$

satisfies the universal property of the inductive limit $\varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_{\mu} \mathcal{T})$. Suppose that $(\mathcal{R}, \{\rho_k\}_{k \in \mathbb{N}})$ is another pair consisting of an operator system and a family of unital completely positive maps $\rho_k : \mathcal{S}_k \otimes_{\mu} \mathcal{T} \rightarrow \mathcal{R}$ such that

$$(33) \quad \rho_{k+1} \circ \psi_k = \rho_k, \quad k \in \mathbb{N}.$$

Suppose that $(\phi_{k,\infty}(s_k), t_1) = (\phi_{l,\infty}(s_l), t_2)$ for some $k, l \in \mathbb{N}$, $s_k \in \mathcal{S}_k$, $s_l \in \mathcal{S}_l$ and $t_1, t_2 \in \mathcal{T}$. By Proposition 3.13, there exists $m > \max\{k, l\}$ such that $\phi_{k,m}(s_k) = \phi_{l,m}(s_l)$. By (33),

$$\begin{aligned} \rho_k(s_k \otimes t_1) &= \rho_m \circ (\phi_{k,m} \otimes \text{id}_{\mathcal{T}})(s_k \otimes t_1) = \rho_m(\phi_{k,m}(s_k) \otimes t_1) \\ &= \rho_m(\phi_{l,m}(s_l) \otimes t_2) = \rho_m \circ (\phi_{l,m} \otimes \text{id}_{\mathcal{T}})(s_l \otimes t_2) = \rho_l(s_l \otimes t_2). \end{aligned}$$

It follows that the map $\theta : (\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \times \mathcal{T} \rightarrow \mathcal{R}$, given by

$$\theta(\phi_{k,\infty}(s_k), t) = \rho_k(s_k \otimes t), \quad k \in \mathbb{N},$$

is well-defined. Clearly, θ is bilinear; let $\tilde{\theta} : (\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T} \rightarrow \mathcal{R}$ be its linearisation. Thus, $\tilde{\theta} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}) = \rho_k$, $k \in \mathbb{N}$. Since ρ_k is unital, $k \in \mathbb{N}$, we have that $\tilde{\theta}$ is unital.

We check that $\tilde{\theta}$ is completely positive. Suppose that $X \in M_n(\mathcal{S}_k \otimes_{\mu} \mathcal{T})$ is such that

$$(\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(X) \in M_n((\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mu} \mathcal{T})^+.$$

By Proposition 4.13, $\phi_{k,\infty}$ is a unital complete order embedding. Since μ is an injective functorial tensor product, $\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}$ is a complete order

embedding. Therefore $X \in M_n(\mathcal{S}_k \otimes_\mu \mathcal{T})^+$ and, since ρ_k is completely positive,

$$\tilde{\theta}^{(n)} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})^{(n)}(X) = \rho_k^{(n)}(X) \in M_n(\mathcal{R})^+.$$

It follows that $\tilde{\theta}$ is completely positive, and the proof is complete. \square

As a direct consequence of Theorem 4.32, we obtain the following fact, which was observed in [24] in the case of complete operator systems.

Corollary 4.33. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **OS** such that each ϕ_k is a complete order isomorphism onto its image, and let \mathcal{T} be an operator system. Then $\varinjlim_{\mathbf{OS}}(\mathcal{S}_k \otimes_{\min} \mathcal{T})$ is unittally completely order isomorphic to $(\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\min} \mathcal{T}$.*

Although the maximal operator system tensor product is not injective, the conclusion of Theorem 4.32 still holds for it, as we show in the next theorem. We note that, in the case where the connecting maps are complete order embeddings, this result was first stated in [23].

Theorem 4.34. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in **OS** and let \mathcal{T} be an operator system. Then $\varinjlim_{\mathbf{OS}}(\mathcal{S}_k \otimes_{\max} \mathcal{T})$ is unittally completely order isomorphic to $(\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\max} \mathcal{T}$.*

Proof. By Proposition 4.29, $\tilde{\alpha} : (\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \odot \mathcal{T} \rightarrow \varinjlim_{\mathbf{OS}}(\mathcal{S}_k \odot \mathcal{T})$ is a linear bijection. Set $D_n = (\tilde{\alpha}^{-1})^{(n)}(M_n(\varinjlim_{\mathbf{OS}}(\mathcal{S}_k \otimes_{\max} \mathcal{T}))^+)$, $n \in \mathbb{N}$. By Lemma 2.8, $\{D_n\}_{n \in \mathbb{N}}$ is an operator system structure on $(\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \odot \mathcal{T}$. We claim that $\{D_n\}_{n \in \mathbb{N}}$ is a tensor product operator system structure. Suppose that $P \in M_p(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)^+$ and $Q \in M_q(\mathcal{T})^+$. For every $r > 0$ there exist $k, l \in \mathbb{N}$, $R \in M_p(\mathcal{S}_l)$, $S \in M_p(\mathcal{S}_k)_h$ and $m > \max\{k, l\}$ such that $\ddot{\phi}_{l,\infty}^{(p)}(R) \in M_p(N)$, $\phi_{k,\infty}^{(n)}(S) = P$ and

$$\frac{r}{\|Q\|} e_m^{(p)} + \phi_{k,m}^{(p)}(S) + \phi_{l,m}^{(p)}(R) \in M_p(\mathcal{S}_m)^+.$$

We have that $S \otimes Q \in M_{pq}(\mathcal{S}_k \otimes_{\max} \mathcal{T})_h$,

$$(\phi_{k,\infty}^{(p)} \otimes \text{id}_{\mathcal{T}}^{(q)})(S \otimes Q) = P \otimes Q$$

and, by Remark 4.30, $\ddot{\psi}_{l,\infty}(R \otimes Q) \in M_{pq}(N_\mu)$. Moreover,

$$\frac{r}{\|Q\|} e_m^{(p)} \otimes Q + (\phi_{k,m} \otimes \text{id}_{\mathcal{T}})^{(pq)}(S \otimes Q) + (\phi_{l,m} \otimes \text{id}_{\mathcal{T}})^{(pq)}(R \otimes Q)$$

belongs to $M_{pq}(\mathcal{S}_m \otimes_{\max} \mathcal{T})^+$, that is,

$$\frac{r}{\|Q\|} e_m^{(p)} \otimes Q + \psi_{k,m}^{(pq)}(S \otimes Q) + \psi_{l,m}^{(pq)}(R \otimes Q) \in M_{pq}(\mathcal{S}_m \otimes_{\max} \mathcal{T})^+.$$

Since $Q \leq \|Q\| e_{\mathcal{T}}^{(q)}$, we conclude that

$$r(e_m^{(p)} \otimes e_{\mathcal{T}}^{(q)}) + \psi_{k,m}^{(pq)}(S \otimes Q) + \psi_{l,m}^{(pq)}(R \otimes Q) \in M_{pq}(\mathcal{S}_m \otimes_{\max} \mathcal{T})^+.$$

Thus,

$$r(e_\infty^{(p)} \otimes e_{\mathcal{T}}^{(q)}) + \psi_{k,\infty}^{(pq)}(S \otimes Q) \in M_{pq}(\varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_{\max} \mathcal{T})^+).$$

Since this holds for every $r > 0$, we have that

$$\psi_{k,\infty}^{(pq)}(S \otimes Q) \in M_{pq}(\varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_{\max} \mathcal{T})^+).$$

However, $\psi_{k,\infty}^{(pq)}(S \otimes Q) = \tilde{\alpha}^{(pq)}(P \otimes Q)$, and we conclude that $P \otimes Q \in D_{pq}$.

Suppose next that $f : \varinjlim \mathbf{OS} \mathcal{S}_k \rightarrow M_p$ and $g : \mathcal{T} \rightarrow M_q$ are unital completely positive maps and that $L \in D_t$, for some $t \in \mathbb{N}$. We will show that $(f \otimes g)^{(t)}(L) \in M_{pqt}^+$, thus obtaining that $\{D_n\}_{n \in \mathbb{N}}$ is an operator system tensor product structure on $(\varinjlim \mathbf{OS} \mathcal{S}_k) \odot \mathcal{T}$. Let $T = \tilde{\alpha}^{(t)}(L)$; we have that $T \in M_t(\varinjlim \mathbf{OS}(\mathcal{S}_k \otimes_{\max} \mathcal{T}))^+$. Fix $r > 0$. Then there exist $k, l \in \mathbb{N}$, $m > \max\{k, l\}$, $R \in M_t(\mathcal{S}_l \odot \mathcal{T})$ and $S \in M_t(\mathcal{S}_k \otimes_{\max} \mathcal{T})_h$ such that $\psi_{k,\infty}^{(t)}(S) = T$, $\psi_{l,\infty}^{(t)}(R) \in M_t(N_{\max})$ and

$$\frac{r}{2}e^{(t)} + \psi_{k,m}^{(t)}(S) + \psi_{l,m}^{(t)}(R) \in M_t(\mathcal{S}_m \otimes_{\max} \mathcal{T})^+.$$

By the definition of the maximal operator system structure, there exist $a, b \in \mathbb{N}$, $A \in M_{ab,t}$, $P \in M_a(\mathcal{S}_m)^+$ and $Q \in M_b(\mathcal{T})^+$ such that

$$re^{(t)} + \psi_{k,m}^{(t)}(S) + \psi_{l,m}^{(t)}(R) = A^*(P \otimes Q)A.$$

The last identity can be rewritten as

$$re^{(t)} + (\phi_{k,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(S) + (\phi_{l,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(R) = A^*(P \otimes Q)A.$$

Note that

$$\tilde{\alpha}^{(t)} \circ (\phi_{m,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})(\phi_{k,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(S) = \psi_{k,\infty}^{(t)}(S) = T = \tilde{\alpha}^{(t)}(L);$$

the injectivity of $\tilde{\alpha}$ implies that

$$(\phi_{k,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})(S) = (\phi_{m,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})(\phi_{k,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(S) = L.$$

Using Remark 4.30, we have that

$$\begin{aligned} & rI_{pqt} + (f \otimes g)^{(t)}(L) \\ &= (f \otimes g)^{(t)}(re^{(t)}) + (f \otimes g)^{(t)} \circ (\phi_{m,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})((\phi_{k,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(S)) \\ &+ (f \otimes g)^{(t)} \circ (\phi_{m,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})(\phi_{l,m}^{(t)} \otimes \text{id}_{\mathcal{T}})(R)) \\ &= (f \otimes g)^{(t)}((\phi_{m,\infty}^{(t)} \otimes \text{id}_{\mathcal{T}})(A^*(P \otimes Q)A)) \\ &= (f \otimes g)^{(t)}(A^*(\phi_{m,\infty}^{(a)}(P) \otimes Q)A) \\ &= A^*(f^{(a)}(\phi_{m,\infty}^{(a)}(P)) \otimes g^{(b)}(Q))A \in M_{pqt}^+. \end{aligned}$$

Suppose \mathcal{H} is a Hilbert space and $\theta : (\varinjlim \mathbf{OS} \mathcal{S}_k) \times \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is a unital jointly completely positive map. Let $\tilde{\theta}$ denote the linearisation of θ . Then $\tilde{\theta} : (\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\max} \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is a unital completely positive map. Since $\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}} : \mathcal{S}_k \otimes_{\max} \mathcal{T} \rightarrow (\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\max} \mathcal{T}$ is a unital completely positive

map, we have that $\tilde{\theta} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}) : \mathcal{S}_k \otimes_{\max} \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is a unital completely positive map, $k \in \mathbb{N}$. Furthermore,

$$\tilde{\theta} \circ (\phi_{k+1,\infty} \otimes \text{id}_{\mathcal{T}}) \circ (\phi_k \otimes \text{id}_{\mathcal{T}}) = \tilde{\theta} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}), \quad k \in \mathbb{N}.$$

By Theorem 4.11, there exists a unique unital completely positive map $\eta : \varinjlim_{\text{OS}} (\mathcal{S}_k \otimes_{\max} \mathcal{T}) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\eta \circ \psi_{k,\infty} = \tilde{\theta} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}})$. Thus,

$$\tilde{\theta} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}) = \eta \circ \psi_{k,\infty} = \eta \circ \tilde{\alpha} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}), \quad k \in \mathbb{N}.$$

Therefore $\tilde{\theta} = \eta \circ \tilde{\alpha}$; that is, $\tilde{\theta} \circ \tilde{\alpha}^{-1} = \eta$. It follows that $\tilde{\theta} \circ \tilde{\alpha}^{-1}$ is a unital completely positive map; that is, $\tilde{\theta}$ is completely positive for the operator system structure $\{D_n\}_{n \in \mathbb{N}}$. By Theorem 2.5, $\tilde{\alpha}$ is a completely positive map. \square

Our next aim is to identify conditions that guarantee that the inductive limit intertwines the commuting tensor product.

Lemma 4.35. *Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e)$ be an operator system and let $\hat{\mathcal{S}}$ be the completion of \mathcal{S} . If \hat{C}_n is the completion of C_n , $n \in \mathbb{N}$ then $(\hat{\mathcal{S}}, \{\hat{C}_n\}_{n \in \mathbb{N}}, e)$ is an operator system. Moreover, if $\rho : \mathcal{S} \rightarrow \mathcal{B}(H)$ is a unital complete isometry then $\hat{\mathcal{S}}$ is unittally completely order isomorphic to the concrete operator system $\overline{\rho(\mathcal{S})}$.*

Proof. Let $\rho : \mathcal{S} \rightarrow \mathcal{B}(H)$ be a unital complete isometry, and let $\mathcal{T} = \overline{\rho(\mathcal{S})}$. We equip \mathcal{T} with the canonical operator system structure arising from its inclusion $\mathcal{T} \subseteq \mathcal{B}(H)$. We claim that $M_n(\mathcal{T})^+ = \overline{M_n(\rho(\mathcal{S}))}^+$, $n \in \mathbb{N}$. It suffices to establish the identity in the case $n = 1$. Suppose that $x \in \mathcal{T}^+$, $r > 0$, and let $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{S}_h$ be a sequence such that $rI + x = \lim_{k \rightarrow \infty} \rho(x_k)$. By [26, Theorem 2], there exists $k_0 \in \mathbb{N}$ such that $x_k \geq 0$, $k \geq k_0$. It follows that $rI + x \in \overline{\rho(\mathcal{S})}^+$, for every $r > 0$. Thus, $x \in \overline{\rho(\mathcal{S})}^+$. The statements of the lemma are now evident. \square

Lemma 4.36. *Let \mathcal{S} and \mathcal{T} be operator systems and let $\hat{\mathcal{S}}$ be the completion of \mathcal{S} . Then $\text{id}_{\mathcal{S}} \otimes \text{id}_{\mathcal{T}} : \mathcal{S} \otimes_{\max} \mathcal{T} \rightarrow \hat{\mathcal{S}} \otimes_{\max} \mathcal{T}$ is a complete order isomorphism onto its image.*

Proof. Fix $n \in \mathbb{N}$ and suppose that $U \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T}) \cap M_n(\hat{\mathcal{S}} \otimes_{\max} \mathcal{T})^+$. Since the set of hermitian elements is closed, $U = U^*$. For all $r > 0$, we have that $r(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + U = \alpha(P^r \otimes Q^r)\alpha^*$ where $\alpha \in M_{n,km}$, $P^r \in M_k(\hat{\mathcal{S}})^+$ and $Q^r \in M_m(\mathcal{T})^+$ for some $k, m \in \mathbb{N}$. By Lemma 4.35, $P^r = \lim_{l \rightarrow \infty} P_l^r$, for some sequence $(P_l^r)_{l \in \mathbb{N}} \subseteq M_n(\mathcal{S})^+$. Let $X_l^r = \alpha(P_l^r \otimes Q^r)\alpha^*$, $l \in \mathbb{N}$. It follows that $r(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + U = \lim_{l \rightarrow \infty} X_l^r$ with $X_l^r \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ for all $r > 0$.

Fix $r > 0$ and choose $l \in \mathbb{N}$ such that

$$\left\| \frac{r}{2}(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + U - X_l^r \right\|_{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})} < \frac{r}{2}.$$

We have

$$\frac{r}{2}(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + \frac{r}{2}(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + U - X_l^{\frac{r}{2}} \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+.$$

Thus $r(e_{\mathcal{S}} \otimes e_{\mathcal{T}})^{(n)} + U \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+$. Since this holds for all $r > 0$ and $M_n(\mathcal{S} \otimes_{\max} \mathcal{T})$ is an AOU space, $U \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+$. \square

In the case the inductive limit is taken in the category of complete operator systems, Theorem 4.38 below follows from [24, Proposition 4.1]. In our proof, we also supply some details that were not fully provided in [24]. First we need a lemma that may be interesting in its own right.

Lemma 4.37. *Let \mathcal{S} and \mathcal{T} be operator systems, and $\iota : \mathcal{S} \otimes_{\mathbf{c}} \mathcal{T} \rightarrow C_u^*(\mathcal{S} \otimes_{\mathbf{c}} \mathcal{T})$ and $j : \mathcal{S} \otimes_{\mathbf{c}} \mathcal{T} \rightarrow C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$ be the canonical embeddings. Then there exists a *-isomorphism $\delta : C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow C_u^*(\mathcal{S} \otimes_{\mathbf{c}} \mathcal{T})$ such that $\delta \circ j = \iota$.*

Proof. Let H be a Hilbert space and $\rho : \mathcal{S} \otimes_{\mathbf{c}} \mathcal{T} \rightarrow \mathcal{B}(H)$ be a unital completely positive map. Let $\rho_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{B}(H)$ and $\rho_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{B}(H)$ be the unital completely positive maps such that $\rho(x \otimes y) = \rho_{\mathcal{S}}(x)\rho_{\mathcal{T}}(y)$, $x \in \mathcal{S}$, $y \in \mathcal{T}$. Let $\tilde{\rho}_{\mathcal{S}} : C_u^*(\mathcal{S}) \rightarrow \mathcal{B}(H)$ and $\tilde{\rho}_{\mathcal{T}} : C_u^*(\mathcal{T}) \rightarrow \mathcal{B}(H)$ be their canonical *-homomorphic extensions. Since the ranges of $\rho_{\mathcal{S}}$ and $\rho_{\mathcal{T}}$ commute, so do the ranges of $\tilde{\rho}_{\mathcal{S}}$ and $\tilde{\rho}_{\mathcal{T}}$. Let $\theta : C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow \mathcal{B}(H)$ be the *-homomorphism given by $\theta(x \otimes y) = \tilde{\rho}_{\mathcal{S}}(x)\tilde{\rho}_{\mathcal{T}}(y)$, $x \in C_u^*(\mathcal{S})$, $y \in C_u^*(\mathcal{T})$. Note that $\theta \circ j = \rho$. Thus, the pair $(C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}), j)$ satisfies the universal property of $C_u^*(\mathcal{S} \otimes_{\mathbf{c}} \mathcal{T})$. The conclusion follows from the uniqueness of the universal C*-algebra. \square

Theorem 4.38. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} such that each ϕ_k is a complete order embedding, and let \mathcal{T} be an operator system. Assume that the map $\phi_k \otimes \text{id}_{\mathcal{T}}$ is a complete order embedding of $\mathcal{S}_k \otimes_{\mathbf{c}} \mathcal{T}$ into $\mathcal{S}_{k+1} \otimes_{\mathbf{c}} \mathcal{T}$, $k \in \mathbb{N}$. Then $\varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_{\mathbf{c}} \mathcal{T})$ is unittally completely order isomorphic to $(\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\mathbf{c}} \mathcal{T}$.*

Proof. Let $\iota_{\mathcal{T}} : \mathcal{T} \rightarrow C_u^*(\mathcal{T})$ and $\iota_k : \mathcal{S}_k \rightarrow C_u^*(\mathcal{S}_k)$, $k \in \mathbb{N}$, be the corresponding canonical embeddings. Consider the following inductive system in \mathbf{C}^* , and therefore in \mathbf{OS} :

$$(34) \quad C_u^*(\mathcal{S}_1) \xrightarrow{\eta_1} C_u^*(\mathcal{S}_2) \xrightarrow{\eta_2} C_u^*(\mathcal{S}_3) \xrightarrow{\eta_3} C_u^*(\mathcal{S}_4) \xrightarrow{\eta_4} \dots,$$

where η_k is the extension of ϕ_k , $k \in \mathbb{N}$, guaranteed by the universal property of the universal C*-algebra. By Lemma 4.22, η_k is a *-isomorphic embedding for all $k \in \mathbb{N}$. Let

$$\eta_{k,\infty} : C_u^*(\mathcal{S}_k) \rightarrow \varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k)$$

be the unital complete order embeddings associated with the inductive system (34). Consider the inductive system

$$(35) \quad C_u^*(\mathcal{S}_1) \otimes_{\max} C_u^*(\mathcal{T}) \xrightarrow{\rho_1} C_u^*(\mathcal{S}_2) \otimes_{\max} C_u^*(\mathcal{T}) \xrightarrow{\rho_2} C_u^*(\mathcal{S}_3) \otimes_{\max} C_u^*(\mathcal{T}) \xrightarrow{\rho_3} \dots$$

in \mathbf{OS} , where $\rho_k = \eta_k \otimes \text{id}_{C_u^*(\mathcal{T})}$, $k \in \mathbb{N}$. By assumption, the map $\phi_k \otimes \text{id}_{\mathcal{T}} : \mathcal{S}_k \otimes_c \mathcal{T} \rightarrow \mathcal{S}_{k+1} \otimes_c \mathcal{T}$ is a complete order isomorphic embedding, $k \in \mathbb{N}$. By Lemmas 4.22 and 4.37, ρ_k is a complete order embedding, $k \in \mathbb{N}$. Let

$$\rho_{k,\infty} : C_u^*(\mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow \varinjlim_{\mathbf{OS}} (C_u^*(\mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T})), \quad k \in \mathbb{N},$$

be the unital completely order isomorphic embeddings associated with the inductive system (35), and let $\tilde{\alpha} : (\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \odot \mathcal{T} \rightarrow \varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_c \mathcal{T})$ be the linear bijection from Proposition 4.29. Note that

$$(36) \quad \tilde{\alpha} \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}) = \psi_{k,\infty}, \quad k \in \mathbb{N},$$

where $\{\psi_{k,\infty}\}_{k \in \mathbb{N}}$ are the unital completely order isomorphic embeddings associated to $\varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_c \mathcal{T})$ (with connecting mappings $\psi_k = \phi_k \otimes \text{id}$, $k \in \mathbb{N}$). Let

$$\tilde{\beta} : (\varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k)) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow \varinjlim_{\mathbf{OS}} (C_u^*(\mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T}))$$

be the unital complete order isomorphism such that

$$(37) \quad \tilde{\beta} \circ (\eta_{k,\infty} \otimes \text{id}_{C_u^*(\mathcal{T})}) = \rho_{k,\infty}, \quad k \in \mathbb{N},$$

whose existence is guaranteed by Theorem 4.34.

Consider the commutative diagram

$$(38) \quad \begin{array}{ccccc} \mathcal{S}_1 \otimes_c \mathcal{T} & \xrightarrow{\psi_1} & \mathcal{S}_2 \otimes_c \mathcal{T} & \xrightarrow{\psi_2} & \dots \\ \iota_1 \otimes \iota_{\mathcal{T}} \downarrow & & \iota_2 \otimes \iota_{\mathcal{T}} \downarrow & & \\ C_u^*(\mathcal{S}_1) \otimes_{\max} C_u^*(\mathcal{T}) & \xrightarrow{\rho_1} & C_u^*(\mathcal{S}_2) \otimes_{\max} C_u^*(\mathcal{T}) & \xrightarrow{\rho_2} & \dots \end{array}$$

and note that all the maps appearing in it are unital complete order embeddings. By Remark 4.15, there exists a unique unital complete order embedding

$$\iota : \varinjlim_{\mathbf{OS}} (\mathcal{S}_k \otimes_c \mathcal{T}) \rightarrow \varinjlim_{\mathbf{OS}} (C_u^*(\mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T}))$$

such that

$$(39) \quad \iota \circ \psi_{k,\infty} = \rho_{k,\infty} \circ (\iota_k \otimes \iota_{\mathcal{T}}), \quad k \in \mathbb{N}.$$

By Lemma 4.36 and Theorem 4.17, the canonical map

$$\gamma : \left(\varinjlim_{\mathbf{OS}} C_u^*(\mathcal{S}_k) \right) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow \left(\varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k) \right) \otimes_{\max} C_u^*(\mathcal{T})$$

is a completely order isomorphic embedding. By Theorem 4.23, there exists a unital *-isomorphism $\mu : \varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k) \rightarrow C_u^*(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)$ such that

$$(40) \quad \mu \circ \eta_{k,\infty} \circ \iota_k = \iota_{\mathcal{S}_{\infty}} \circ \phi_{k,\infty}$$

for all $k \in \mathbb{N}$, where $\iota_{\mathcal{S}_{\infty}} : \varinjlim_{\mathbf{OS}} \mathcal{S}_k \rightarrow C_u^*(\varinjlim_{\mathbf{OS}} \mathcal{S}_k)$ is the canonical embedding. We have that

$$\mu \otimes \text{id}_{C_u^*(\mathcal{T})} : \left(\varinjlim_{\mathbf{C}^*} C_u^*(\mathcal{S}_k) \right) \otimes_{\max} C_u^*(\mathcal{T}) \rightarrow C_u^*(\varinjlim_{\mathbf{OS}} \mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T})$$

is a unital $*$ -isomorphism. By the definition of the commuting tensor product,

$$\iota_{\mathcal{S}_\infty} \otimes \iota_{\mathcal{T}} : (\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\mathbf{c}} \mathcal{T} \rightarrow C_u^*(\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\max} C_u^*(\mathcal{T})$$

is a unital complete order isomorphism onto its image.

We will show that

$$(41) \quad (\iota_{\mathcal{S}_\infty} \otimes \iota_{\mathcal{T}}) \circ \tilde{\alpha}^{-1} = (\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ \gamma \circ \tilde{\beta}^{-1} \circ \iota;$$

since $(\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ \gamma \circ \tilde{\beta}^{-1} \circ \iota$ and $\iota_{\mathcal{S}_\infty} \otimes \iota_{\mathcal{T}}$ are complete order embeddings, it will follow from Lemma 2.9 that $\tilde{\alpha}$ is a complete order embedding. By (36), (37), (39) and (40), for every $k \in \mathbb{N}$, we have

$$\begin{aligned} & (\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ \gamma \circ \tilde{\beta}^{-1} \circ \iota \circ \psi_{k,\infty} \\ &= (\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ \tilde{\beta}^{-1} \circ \iota \circ \psi_{k,\infty} \\ &= (\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ \tilde{\beta}^{-1} \circ \rho_{k,\infty} \circ (\iota_k \otimes \iota_{\mathcal{T}}) \\ &= (\mu \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ (\eta_{k,\infty} \otimes \text{id}_{C_u^*(\mathcal{T})}) \circ (\iota_k \otimes \iota_{\mathcal{T}}) \\ &= (\mu \circ \eta_{k,\infty} \circ \iota_k) \otimes (\text{id}_{C_u^*(\mathcal{T})} \circ \iota_{\mathcal{T}}) = (\iota_{\mathcal{S}_\infty} \circ \phi_{k,\infty}) \otimes \iota_{\mathcal{T}} \\ &= (\iota_{\mathcal{S}_\infty} \otimes \iota_{\mathcal{T}}) \circ (\phi_{k,\infty} \otimes \text{id}_{\mathcal{T}}) = (\iota_{\mathcal{S}_\infty} \otimes \iota_{\mathcal{T}}) \circ \tilde{\alpha}^{-1} \circ \psi_{k,\infty}. \end{aligned}$$

This establishes (41), and the proof is complete. \square

Remark Recall [19] that an operator system \mathcal{S} is said to possess the *double commutant expectation property (DCEP)* if, for every complete order embedding $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ (where \mathcal{H} is a Hilbert space), there exists a completely positive map from $\mathcal{B}(\mathcal{H})$ into the double commutant \mathcal{S}'' of \mathcal{S} that fixes \mathcal{S} element-wise. By [19, Theorem 7.1], if \mathcal{S}_k satisfies DCEP for each $k \in \mathbb{N}$ then the assumption in Theorem 4.38 is automatically satisfied, and hence its conclusion holds true.

We finish this section with an application of Theorem 4.38.

Theorem 4.39. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} such that each ϕ_k is a complete order embedding. If \mathcal{S}_k has the DCEP for each $k \in \mathbb{N}$, then so does $\varinjlim \mathbf{OS} \mathcal{S}_k$.*

Proof. Let \mathcal{T} be an operator system and \mathcal{R} be an operator system with $\varinjlim \mathbf{OS} \mathcal{S}_k \subseteq_{\text{coi}} \mathcal{R}$. By Remark 4.15 and [19, Theorems 7.1 and 7.3],

$$(42) \quad \mathcal{S}_k \otimes_{\mathbf{c}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{S}_{k+1} \otimes_{\mathbf{c}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{R} \otimes_{\mathbf{c}} \mathcal{T},$$

for every $k \in \mathbb{N}$. By (42) and Remark 4.15,

$$\varinjlim \mathbf{OS} (\mathcal{S}_k \otimes_{\mathbf{c}} \mathcal{T}) \subseteq_{\text{coi}} \mathcal{R} \otimes_{\mathbf{c}} \mathcal{T}.$$

On the other hand, by Theorem 4.38, $\varinjlim \mathbf{OS} (\mathcal{S}_k \otimes_{\mathbf{c}} \mathcal{T}) = (\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\mathbf{c}} \mathcal{T}$. It follows that

$$(\varinjlim \mathbf{OS} \mathcal{S}_k) \otimes_{\mathbf{c}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{R} \otimes_{\mathbf{c}} \mathcal{T}.$$

By [19, Theorems 7.1 and 7.3], $\varinjlim \mathbf{OS} \mathcal{S}_k$ has the DCEP. \square

5. INDUCTIVE LIMITS OF OPERATOR C*-SYSTEMS

In this section, we adapt our construction of the inductive limit of operator systems to the category of operator C*-systems. We recall some notions and results that will be required shortly. Let $(\mathcal{S}, \{C_n\}_{n \in \mathbb{N}}, e)$ be a complete operator system and \mathcal{A} be a unital C*-algebra such that \mathcal{S} is a completely contractive \mathcal{A} -bimodule. Let us denote the bimodule action by \cdot so that $(a_1 a_2) \cdot s = a_1 \cdot (a_2 \cdot s)$ whenever $s \in \mathcal{S}$ and $a_1, a_2 \in \mathcal{A}$. We assume that $a \cdot e = e \cdot a$, $a \in \mathcal{A}$, and equip $M_n(\mathcal{S})$ with a bimodule action of $M_n(\mathcal{A})$ by letting $(a_{i,j}) \cdot (s_{i,j}) = (\sum_{k=1}^n a_{i,k} \cdot s_{k,j})$ and $(s_{i,j}) \cdot (a_{i,j}) = (\sum_{k=1}^n s_{i,k} \cdot a_{k,j})$. If

$$A^* \cdot X \cdot A \in C_n \text{ whenever } X \in C_n,$$

we say that \mathcal{S} is an *operator \mathcal{A} -system* or that the pair $(\mathcal{S}, \mathcal{A})$ is an *operator C*-system*. Let $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{T}, \mathcal{B})$ be operator C*-systems. A pair (ϕ, π) will be called an *operator C*-system homomorphism* if $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map, $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is a unital *-homomorphism and $\phi(a_1 \cdot s \cdot a_2) = \pi(a_1) \cdot \phi(s) \cdot \pi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$ and $s \in \mathcal{S}$. We write $(\phi, \pi) : (\mathcal{S}, \mathcal{A}) \rightarrow (\mathcal{T}, \mathcal{B})$. We call the operator C*-system homomorphism (ϕ, π) an *operator C*-system monomorphism* if ϕ is completely isometric. If $(\phi, \pi) : (\mathcal{S}, \mathcal{A}) \rightarrow (\mathcal{T}, \mathcal{B})$ and $(\psi, \rho) : (\mathcal{T}, \mathcal{B}) \rightarrow (\mathcal{R}, \mathcal{C})$ are operator C*-system homomorphisms, we write $(\phi, \pi) \circ (\psi, \rho)$ for the pair $(\phi \circ \psi, \pi \circ \rho)$; it is straightforward to see that the latter is an operator C*-system homomorphism. The following theorem is contained in [29, Chapter 15].

Theorem 5.1. *Let $(\mathcal{S}, \mathcal{A})$ be an operator C*-system. Then there exists a Hilbert space \mathcal{H} and an operator C*-system monomorphism $(\Phi, \Pi) : (\mathcal{S}, \mathcal{A}) \rightarrow (\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ such that the order unit of \mathcal{S} is mapped to the identity operator.*

We denote by $\mathbf{OC}^*\mathbf{S}$ the category whose objects are operator C*-systems and whose morphisms are operator C*-system homomorphisms.

Before considering inductive systems in $\mathbf{OC}^*\mathbf{S}$, we make some observations which we shall refer to later in the section. The proof of the following lemma is straightforward and is omitted.

Lemma 5.2. *Let $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ be an inductive system in \mathbf{OS} . If $s_{k_n} \in \mathcal{S}_{k_n}$ and $(\phi_{k_n, \infty}(s_{k_n}))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\varinjlim_{\mathbf{OS}} \mathcal{S}_k$ then there exists a sequence $(m_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that $(\phi_{k_n, m_n}(s_{k_n}))_{n \in \mathbb{N}}$ is a bounded sequence.*

We fix throughout the section an inductive system

$$(43) \quad (\mathcal{S}_1, \mathcal{A}_1) \xrightarrow{(\phi_1, \pi_1)} (\mathcal{S}_2, \mathcal{A}_2) \xrightarrow{(\phi_2, \pi_2)} (\mathcal{S}_3, \mathcal{A}_3) \xrightarrow{(\phi_3, \pi_3)} (\mathcal{S}_4, \mathcal{A}_4) \xrightarrow{(\phi_4, \pi_4)} \dots$$

in $\mathbf{OC}^*\mathbf{S}$. Thus, $\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \mathcal{S}_4 \xrightarrow{\phi_4} \dots$ is an inductive system in \mathbf{OS} , $\mathcal{A}_1 \xrightarrow{\pi_1} \mathcal{A}_2 \xrightarrow{\pi_2} \mathcal{A}_3 \xrightarrow{\pi_3} \mathcal{A}_4 \xrightarrow{\pi_4} \dots$ is an inductive system in \mathbf{C}^* , \mathcal{S}_k is an

operator \mathcal{A}_k -system and (ϕ_k, π_k) is an operator C^* -system homomorphism, $k \in \mathbb{N}$. We set

$$\mathcal{S}_\infty = \varinjlim \mathcal{S}_k, \mathcal{A}_\infty = \varinjlim \mathcal{A}_k, \widehat{\mathcal{A}}_\infty = \varinjlim_{C^*} \mathcal{A}_k,$$

and $\widehat{\mathcal{S}}_\infty$ to be the completion of \mathcal{S}_∞ . By Theorem 4.17, $\widehat{\mathcal{A}}_\infty$ is the completion of \mathcal{A}_∞ .

We proceed with the construction of the inductive limit of (43). Let $a \in \widehat{\mathcal{A}}_\infty$ and $s \in \widehat{\mathcal{S}}_\infty$. Then $a = \lim_{n \rightarrow \infty} \pi_{k_n, \infty}(a_{k_n})$ and $s = \lim_{n \rightarrow \infty} \phi_{l_n, \infty}(s_{l_n})$ for some $a_{k_n} \in \mathcal{A}_{k_n}$ and $s_{l_n} \in \mathcal{S}_{l_n}$. Letting $m_n = \max\{k_n, l_n\}$, $a_{m_n} = \pi_{k_n, m_n}(a_{k_n})$ and $s_{m_n} = \phi_{l_n, m_n}(s_{l_n})$, we have

$$a = \lim_{n \rightarrow \infty} \pi_{m_n, \infty}(a_{m_n}) \quad \text{and} \quad s = \lim_{n \rightarrow \infty} \phi_{m_n, \infty}(s_{m_n}).$$

We let

$$(44) \quad a \cdot s = \lim_{n \rightarrow \infty} \phi_{m_n, \infty}(a_{m_n} \cdot s_{m_n}) \quad \text{and} \quad s \cdot a = \lim_{n \rightarrow \infty} \phi_{m_n, \infty}(s_{m_n} \cdot a_{m_n}).$$

It is straightforward to check that the operations (44) are well-defined and turn $\widehat{\mathcal{S}}_\infty$ into a completely contractive $\widehat{\mathcal{A}}_\infty$ -bimodule.

Remark 5.3. Note that, if $k \in \mathbb{N}$, $a, b \in \mathcal{A}_k$ and $s \in \mathcal{S}_k$ then

$$\pi_{k, \infty}(a) \cdot \phi_{k, \infty}(s) \cdot \pi_{k, \infty}(b) = \phi_{k, \infty}(a \cdot s \cdot b).$$

Lemma 5.4. If $S \in M_n(\mathcal{S}_\infty)^+$ and $A \in M_n(\mathcal{A}_\infty)$ then $A^* \cdot S \cdot A \in M_n(\mathcal{S}_\infty)^+$.

Proof. Write $S = \phi_{k, \infty}^{(n)}(S_k) \in M_n(\mathcal{S}_\infty)^+$ and $A = \pi_{k, \infty}^{(n)}(A_k) \in M_n(\mathcal{A}_\infty)$, where $S_k \in M_n(\mathcal{S}_k)$ and $A_k \in M_n(\mathcal{A}_k)$ for some k . Then $A_k^* \cdot S_k \cdot A_k \in M_n(\mathcal{S}_k)^+$. Since the map $\phi_{k, \infty}$ is completely positive, Remark 5.3 shows that

$$\begin{aligned} A^* \cdot S \cdot A &= \pi_{k, \infty}^{(n)}(A_k^*) \cdot \phi_{k, \infty}^{(n)}(S_k) \cdot \pi_{k, \infty}^{(n)}(A_k) \\ &= \phi_{k, \infty}^{(n)}(A_k^* \cdot S_k \cdot A_k) \in M_n(\mathcal{S}_\infty)^+. \end{aligned}$$

□

Proposition 5.5. The space $\widehat{\mathcal{S}}_\infty$ is an operator $\widehat{\mathcal{A}}_\infty$ -system and $(\phi_{k, \infty}, \pi_{k, \infty})$ is an operator C^* -system homomorphism from $(\mathcal{S}_k, \mathcal{A}_k)$ into $(\widehat{\mathcal{S}}_\infty, \widehat{\mathcal{A}}_\infty)$ such that $(\phi_{k+1, \infty}, \pi_{k+1, \infty}) \circ (\phi_k, \pi_k) = (\phi_{k, \infty}, \pi_{k, \infty})$, $k \in \mathbb{N}$.

Proof. it is clear that $\widehat{\mathcal{S}}_\infty$ is a complete operator system. Suppose $S \in M_n(\widehat{\mathcal{S}}_\infty)^+$ and $A \in M_n(\widehat{\mathcal{A}}_\infty)$ so that $S = \lim_{p \rightarrow \infty} S_p$ and $A = \lim_{p \rightarrow \infty} A_p$ where $S_p \in M_n(\mathcal{S}_\infty)^+$ and $A_p \in M_n(\mathcal{A}_\infty)$. Then $A^* \cdot S \cdot A = \lim_{p \rightarrow \infty} A_p^* \cdot S_p \cdot A_p$ and, by Lemma 5.4, $A_p^* \cdot S_p \cdot A_p \in M_n(\mathcal{S}_\infty)^+$ for all $p \in \mathbb{N}$. Since the cone $M_n(\widehat{\mathcal{S}}_\infty)^+$ is closed, $A^* \cdot S \cdot A \in M_n(\widehat{\mathcal{S}}_\infty)^+$. □

Theorem 5.6. The triple $(\widehat{\mathcal{S}}_\infty, \widehat{\mathcal{A}}_\infty, \{\phi_{k, \infty}, \pi_{k, \infty}\}_{k \in \mathbb{N}})$ is an inductive limit of the inductive system

$$(\mathcal{S}_1, \mathcal{A}_1) \xrightarrow{(\phi_1, \pi_1)} (\mathcal{S}_2, \mathcal{A}_2) \xrightarrow{(\phi_2, \pi_2)} (\mathcal{S}_3, \mathcal{A}_3) \xrightarrow{(\phi_3, \pi_3)} (\mathcal{S}_4, \mathcal{A}_4) \xrightarrow{(\phi_4, \pi_4)} \dots$$

in $\mathbf{OC}^*\mathbf{S}$.

Proof. Suppose $((\mathcal{T}, \mathcal{B}), \{(\psi_k, \rho_k)\}_{k \in \mathbb{N}})$ is a pair consisting of a complete operator C*-system and a family of operator C*-system homomorphisms $(\psi_k, \rho_k) : (\mathcal{S}_k, \mathcal{A}_k) \rightarrow (\mathcal{T}, \mathcal{B})$ such that $(\psi_{k+1}, \rho_{k+1}) \circ (\phi_k, \pi_k) = (\psi_k, \rho_k)$ for all $k \in \mathbb{N}$. By Theorem 4.11, there exists a unique unital completely positive map $\psi : \mathcal{S}_\infty \rightarrow \mathcal{T}$ such that $\psi \circ \phi_{k,\infty} = \psi_k$. Let $\hat{\psi} : \hat{\mathcal{S}}_\infty \rightarrow \mathcal{T}$ be the continuous extension of ψ . Lemma 4.35 easily implies that $\hat{\psi}$ is completely positive. By Section 2.5, there exists a unique unital *-homomorphism $\hat{\rho} : \varinjlim_{\mathbf{C}^*} \mathcal{A}_k \rightarrow \mathcal{B}$ such that $\hat{\rho} \circ \pi_{k,\infty} = \rho_k$. A direct verification shows that $\hat{\psi}(a \cdot s \cdot b) = \hat{\rho}(a) \cdot \hat{\psi}(s) \cdot \hat{\rho}(b)$. \square

We denote the inductive limit whose existence is established in Theorem 5.6 by $\varinjlim_{\mathbf{OC}^* \mathbf{S}} (\mathcal{S}_k, \mathcal{A}_k)$ or $\varinjlim_{\mathbf{OC}^* \mathbf{S}} \mathcal{S}_k$, when the context is clear.

Remark 5.7. Let $(\{\mathcal{S}_k, \mathcal{A}_k\}_{k \in \mathbb{N}}, \{\phi_k, \pi_k\}_{k \in \mathbb{N}})$ and $(\{\mathcal{T}_k, \mathcal{B}_k\}_{k \in \mathbb{N}}, \{\psi_k, \rho_k\}_{k \in \mathbb{N}})$ be inductive systems in $\mathbf{OC}^* \mathbf{S}$ and let $\{(\theta_k, \varphi_k)\}_{k \in \mathbb{N}}$ be a sequences of operator C*-system homomorphisms such that the following diagrams commute:

$$\begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 \xrightarrow{\phi_4} \dots \\ \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow \\ \mathcal{T}_1 & \xrightarrow{\psi_1} & \mathcal{T}_2 & \xrightarrow{\psi_2} & \mathcal{T}_3 & \xrightarrow{\psi_3} & \mathcal{T}_4 \xrightarrow{\psi_4} \dots \end{array}$$

and

$$\begin{array}{ccccccc} \mathcal{A}_1 & \xrightarrow{\pi_1} & \mathcal{A}_2 & \xrightarrow{\pi_2} & \mathcal{A}_3 & \xrightarrow{\pi_3} & \mathcal{A}_4 \xrightarrow{\pi_4} \dots \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \varphi_4 \downarrow \\ \mathcal{B}_1 & \xrightarrow{\rho_1} & \mathcal{B}_2 & \xrightarrow{\rho_2} & \mathcal{B}_3 & \xrightarrow{\rho_3} & \mathcal{B}_4 \xrightarrow{\rho_4} \dots \end{array}$$

It follows from Theorem 2.14 and Theorem 5.6 that there exists a unique operator C*-system homomorphism

$$(\hat{\theta}, \hat{\varphi}) : (\varinjlim_{\mathbf{OC}^* \mathbf{S}} \mathcal{S}_k, \varinjlim_{\mathbf{C}^*} \mathcal{A}_k) \rightarrow (\varinjlim_{\mathbf{OC}^* \mathbf{S}} \mathcal{T}_k, \varinjlim_{\mathbf{C}^*} \mathcal{B}_k)$$

such that $(\hat{\theta}, \hat{\varphi}) \circ (\phi_{k,\infty}, \pi_{k,\infty}) = (\psi_{k,\infty}, \rho_{k,\infty}) \circ (\theta_k, \varphi_k)$ for all $k \in \mathbb{N}$.

Remark 5.8. Suppose that $(\{\mathcal{S}_k, \mathcal{A}_k\}_{k \in \mathbb{N}}, \{\phi_k, \pi_k\}_{k \in \mathbb{N}})$ and $(\{\mathcal{T}_k, \mathcal{B}_k\}_{k \in \mathbb{N}}, \{\psi_k, \rho_k\}_{k \in \mathbb{N}})$ are inductive systems in $\mathbf{OC}^* \mathbf{S}$, and let $\{(\theta_{m_k}, \varphi_{m_k})\}_{k \in \mathbb{N}}$ and $\{(\mu_{n_k}, \nu_{n_k})\}_{k \in \mathbb{N}}$ be sequences of operator C*-system monomorphisms such that the diagrams

$$\begin{array}{ccccccc} \mathcal{S}_1 & \xrightarrow{\phi_{1,m_1}} & \mathcal{S}_{m_1} & \xrightarrow{\phi_{m_1,m_2}} & \mathcal{S}_{m_2} & \xrightarrow{\phi_{m_2,m_3}} & \dots \\ \downarrow \theta_1 & \nearrow \mu_{n_1} & \downarrow \theta_{m_1} & \nearrow \mu_{n_2} & \downarrow \theta_{m_2} & \nearrow \mu_{n_3} & \\ \mathcal{T}_{n_1} & \xrightarrow{\psi_{n_1,n_2}} & \mathcal{T}_{n_2} & \xrightarrow{\psi_{n_2,n_3}} & \mathcal{T}_{n_3} & \xrightarrow{\psi_{n_3,n_4}} & \dots \end{array}$$

and

$$\begin{array}{ccccccc} \mathcal{A}_1 & \xrightarrow{\pi_{1,m_1}} & \mathcal{A}_{m_1} & \xrightarrow{\pi_{m_1,m_2}} & \mathcal{A}_{m_2} & \xrightarrow{\pi_{m_2,m_3}} & \dots \\ \downarrow \varphi_1 & \nearrow \nu_{n_1} & \downarrow \varphi_{m_1} & \nearrow \nu_{n_2} & \downarrow \varphi_{m_2} & \nearrow \nu_{n_3} & \\ \mathcal{B}_{n_1} & \xrightarrow{\rho_{n_1,n_2}} & \mathcal{B}_{n_2} & \xrightarrow{\rho_{n_2,n_3}} & \mathcal{B}_{n_3} & \xrightarrow{\rho_{n_3,n_4}} & \dots \end{array}$$

commute. By Theorem 5.6, $\varinjlim_{\mathbf{OC}^*\mathbf{S}}(\mathcal{S}_k, \mathcal{A}_k)$ and $\varinjlim_{\mathbf{OC}^*\mathbf{S}}(\mathcal{T}_k, \mathcal{B}_k)$ are isomorphic. In particular, $\varinjlim_{\mathbf{OC}^*\mathbf{S}} \mathcal{S}_k$ is unitaly completely order isomorphic to $\varinjlim_{\mathbf{OC}^*\mathbf{S}} \mathcal{T}_k$ and $\varinjlim_{\mathbf{C}^*} \mathcal{A}_k$ is unitaly *-isomorphic to $\varinjlim_{\mathbf{C}^*} \mathcal{B}_k$.

6. INDUCTIVE LIMITS OF GRAPH OPERATOR SYSTEMS

In this section, we examine inductive limits of graph operator systems, viewing them as the operator systems of topological graphs *via* the theory of topological equivalence relations [34]. We identify the C*-envelope of such an operator system, and prove an isomorphism theorem; these can be viewed as a topological version of recent results from [28]. We also establish a version of the Glimm Theorem for this class of operator systems. As our results rely crucially on [34] (and thus on [35], [36], [37] and [10]), for the convenience of the reader, we often provide the background and details.

A *UHF algebra* [16] (or, otherwise, *uniformly hyper-finite C*-algebra*) is a C*-algebra which is (*-isomorphic to) the inductive limit of an inductive system

$$(45) \quad M_{n_1} \xrightarrow{\pi_1} M_{n_2} \xrightarrow{\pi_2} M_{n_3} \xrightarrow{\pi_3} M_{n_4} \xrightarrow{\pi_4} \dots,$$

where π_k is a unital *-homomorphism, $k \in \mathbb{N}$. UHF algebras and their classification appear extensively in the literature, see for example [9], [27] or [38]. For each $k \in \mathbb{N}$, let $e_{i,j}^k$ denote the matrix in M_{n_k} with 1 at the $(i,j)^{\text{th}}$ entry and 0 elsewhere and let $l_k = \frac{n_{k+1}}{n_k}$. We have that

$$\pi_k(e_{i,j}^k) = \sum_{r=0}^{l_k-1} e_{rn_k+i, rn_k+j}^{k+1}.$$

We call $e_{i,j}^k$ the *canonical matrix units*.

Let \mathcal{A} be a C*-algebra. A C*-subalgebra of \mathcal{A} is called a *maximal abelian self-adjoint algebra* (*masa*, for short) if it is abelian and not properly contained in another abelian C*-subalgebra of \mathcal{A} . Let

$$\mathcal{D}_1 \xrightarrow{\pi_1} \mathcal{D}_2 \xrightarrow{\pi_2} \mathcal{D}_3 \xrightarrow{\pi_3} \mathcal{D}_4 \xrightarrow{\pi_4} \dots$$

be the inductive system in \mathbf{C}^* induced by (45), where \mathcal{D}_k is the subalgebra of diagonal matrices in M_{n_k} for each $k \in \mathbb{N}$. A proof of the following result may be found in [34, Proposition 4.1].

Proposition 6.1. *The C*-algebra $\varinjlim_{\mathbf{C}^*} \mathcal{D}_k$ is a masa in $\varinjlim_{\mathbf{C}^*} M_{n_k}$.*

Denote by $\Delta(\mathcal{C})$ the Gelfand spectrum of an abelian C*-algebra \mathcal{C} . We call $\varinjlim_{\mathbf{C}^*} \mathcal{D}_k$ the *canonical masa* in the UHF algebra $\varinjlim_{\mathbf{C}^*} M_{n_k}$. Since $\varinjlim_{\mathbf{C}^*} \mathcal{D}_k$ is an abelian C*-algebra, we have that $\varinjlim_{\mathbf{C}^*} \mathcal{D}_k$ is *-isomorphic to $C(X_\infty)$

where $X_\infty = \Delta(\varinjlim_{\mathbf{C}^*} \mathcal{D}_k)$. For the following remark, which is a special case of Remark 2.19, let $X_k = \Delta(\mathcal{D}_k)$.

Remark 6.2. *The space X_∞ is homeomorphic to $\varinjlim_{\mathbf{Top}} X_k$.*

The following theorem, whose proof may be found in [16] (see [27] for an alternative proof), characterises UHF algebras.

Theorem 6.3 (Glimm). *The UHF algebras $\varinjlim_{\mathbf{C}^*} M_{n_k}$ and $\varinjlim_{\mathbf{C}^*} M_{m_k}$ are $*$ -isomorphic if and only if for all $w \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that $n_w | m_x$, and for all $y \in \mathbb{N}$ there exists $z \in \mathbb{N}$ such that $m_y | n_z$.*

Let X be a topological space. We define a *graph* to be a pair $G = (X, E)$ of sets such that $E \subseteq X \times X$ is a closed subset which is symmetric (that is, $(x, y) \in E$ if and only if $(y, x) \in E$) and anti-reflexive (that is, $(x, x) \notin E$ for all $x \in X$). We call the elements of X the *vertices* of G and say that two vertices $x, y \in X$ are *adjacent* if $(x, y) \in E$. Given G , we set $\tilde{G} = (X, \tilde{E})$ where $\tilde{E} = E \cup \{(x, x) : x \in X\}$ is the *extended edge set* of G . Two graphs $G = (X, E)$ and $G' = (X', E')$ are called *isomorphic* if there exists a homeomorphism $\varphi : X \rightarrow X'$ such that $(x, y) \in E$ if and only if $(\varphi(x), \varphi(y)) \in E'$.

Let G be a graph on n vertices so that $X = \{1, \dots, n\}$. Denote by $e_{i,j}$ the $n \times n$ matrix with 1 in its (i, j) -th-entry and 0 elsewhere. We define the *operator system* \mathcal{S}_G of G by letting

$$\mathcal{S}_G = \text{span}\{e_{i,j} : (i, j) \in \tilde{E}\}.$$

A *graph operator system* is an operator system of the form \mathcal{S}_G .

Denote temporarily by \mathcal{D} be the subalgebra of diagonal matrices in M_n . Clearly, $(\mathcal{S}_G, \mathcal{D})$ is an operator \mathbf{C}^* -system when we take the module operation to be the usual matrix multiplication in M_n . The following characterisation is well-known, see [31].

Proposition 6.4. *Let \mathcal{S} be an operator subsystem of M_n . Then \mathcal{S} is a graph operator system if and only if $\mathcal{D}\mathcal{S}\mathcal{D} \subseteq \mathcal{S}$. In this case the graph $G = (X, E)$ is defined by letting $X = \{1, \dots, n\}$ and $E = \{(i, j) : i \neq j \text{ and } e_{i,j} \in \mathcal{S}\}$.*

The following two results about graph operator systems were proved in [28, Theorem 3.2 and Theorem 3.3].

Theorem 6.5 (Paulsen–Ortiz). *Let G be a graph on n vertices. Then the \mathbf{C}^* -subalgebra of M_n generated by \mathcal{S}_G is the \mathbf{C}^* -envelope of \mathcal{S}_G .*

Theorem 6.6 (Paulsen–Ortiz). *Let G_1 and G_2 be graphs on n vertices. Then G_1 is isomorphic to G_2 if and only if \mathcal{S}_{G_1} is unittally completely order isomorphic to \mathcal{S}_{G_2} .*

6.1. Operator \mathbf{C}^* -systems in UHF algebras. We define a *concrete operator \mathbf{C}^* -system* to be a triple $(\mathcal{D}, \mathcal{S}, \mathcal{A})$ where $\mathcal{D}, \mathcal{A} \in \mathbf{C}^*$, $\mathcal{S} \in \mathbf{OS}$, $(\mathcal{S}, \mathcal{D}) \in \mathbf{OC}^*\mathbf{S}$, $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{D}\mathcal{S}\mathcal{D} \subseteq \mathcal{S}$. When the context is clear,

we simplify the notation and call \mathcal{S} a *concrete operator \mathcal{D} -system*, without mention of \mathcal{A} .

Throughout this chapter, we fix an inductive system

$$M_{n_1} \xrightarrow{\pi_1} M_{n_2} \xrightarrow{\pi_2} M_{n_3} \xrightarrow{\pi_3} M_{n_4} \xrightarrow{\pi_4} \dots$$

in \mathbf{C}^* . Suppose that G_k is a graph on n_k vertices, such that $\pi_k(\mathcal{S}_{G_k}) \subseteq \mathcal{S}_{G_{k+1}}$, and let $\phi_k = \pi_k|_{\mathcal{S}_{G_k}}$, $k \in \mathbb{N}$. We thus have inductive systems

$$\mathcal{S}_{G_1} \xrightarrow{\phi_1} \mathcal{S}_{G_2} \xrightarrow{\phi_2} \mathcal{S}_{G_3} \xrightarrow{\phi_3} \mathcal{S}_{G_4} \xrightarrow{\phi_4} \dots$$

and

$$\mathcal{D}_1 \xrightarrow{\pi_1} \mathcal{D}_2 \xrightarrow{\pi_2} \mathcal{D}_3 \xrightarrow{\pi_3} \mathcal{D}_4 \xrightarrow{\pi_4} \dots;$$

since \mathcal{S}_{G_k} is an operator \mathcal{D}_k -system, the latter inductive systems can be viewed as an inductive system in $\mathbf{OC}^*\mathbf{S}$. Note that the inductive limit $\varinjlim_{\mathbf{OC}^*\mathbf{S}} \mathcal{S}_{G_k}$ is the completion of $\varinjlim_{\mathbf{OS}} \mathcal{S}_{G_k}$ or, equivalently, the closure of $\varinjlim_{\mathbf{OS}} \mathcal{S}_{G_k}$ in $\varinjlim_{\mathbf{C}^*} \mathcal{A}_k$. (Here, and in the sequel, write $\mathcal{A}_k = \pi_{k,\infty}(M_{n_k})$; note that $\mathcal{A}_k \cong M_{n_k}$.) We will see that every concrete operator $(\varinjlim_{\mathbf{C}^*} \mathcal{D}_k)$ -system (defined shortly) is the inductive limit of a sequence of graph operator systems, and will associate to $\varinjlim_{\mathbf{OC}^*\mathbf{S}} \mathcal{S}_{G_k}$ a graph which is related to the sequence of graphs $(G_k)_{k \in \mathbb{N}}$.

We will use the following notation to denote the inductive limits:

$$\begin{aligned} \mathcal{S}_\infty &= \varinjlim_{\mathbf{OS}} \mathcal{S}_k, \\ \widehat{\mathcal{S}}_\infty &= \varinjlim_{\mathbf{OC}^*\mathbf{S}} \mathcal{S}_k, \\ \widehat{\mathcal{D}}_\infty &= \varinjlim_{\mathbf{C}^*} \mathcal{D}_k \text{ and} \\ \widehat{\mathcal{A}}_\infty &= \varinjlim_{\mathbf{C}^*} \mathcal{A}_k. \end{aligned}$$

Observe that $(\widehat{\mathcal{D}}_\infty, \widehat{\mathcal{S}}_\infty, \widehat{\mathcal{A}}_\infty)$ is a concrete operator \mathbf{C}^* -system. Since each π_k is a unital injective $*$ -homomorphism, by Remark 2.18, $\pi_{k,\infty}$ is a unital injective $*$ -homomorphism for all $k \in \mathbb{N}$; we therefore sometimes simplify the notation and write a_k in the place of $\pi_{k,\infty}(a_k)$.

Recall [34] that a closed linear subspace \mathcal{S} of $\widehat{\mathcal{A}}_\infty$ is said to be *inductive* relative to $(\mathcal{A}_k)_{k \in \mathbb{N}}$ if

$$\mathcal{S} = \overline{\bigcup_{k \in \mathbb{N}} \mathcal{S} \cap \mathcal{A}_k}.$$

We note the following fact which follows from [34, Theorem 4.7].

Proposition 6.7. *Let $\mathcal{S} \subseteq \widehat{\mathcal{A}}_\infty$ be a concrete operator $\widehat{\mathcal{D}}_\infty$ -system and set $\mathcal{S}_k = \mathcal{S} \cap \mathcal{A}_k$. Then $\mathcal{S}_k \subseteq \mathcal{A}_k$ is a concrete operator \mathcal{D}_k -system and $\mathcal{S} = \varinjlim_{\mathbf{OS}} \mathcal{S}_k$.*

The next result is an infinite dimensional analogue of Theorem 6.5.

Theorem 6.8. *Let $\widehat{\mathcal{S}}_\infty \subseteq \widehat{\mathcal{A}}_\infty$ be a concrete operator $\widehat{\mathcal{D}}_\infty$ -system. The \mathbf{C}^* -envelope of $\widehat{\mathcal{S}}_\infty$ coincides with the \mathbf{C}^* -subalgebra of $\widehat{\mathcal{A}}_\infty$ generated by $\widehat{\mathcal{S}}_\infty$.*

Proof. Let $C^*(\widehat{\mathcal{S}}_\infty)$ denote the C^* -subalgebra of $\widehat{\mathcal{A}}_\infty$ generated by the operator system $\widehat{\mathcal{S}}_\infty$ and let $C^*(\mathcal{S}_k)$ denote the C^* -subalgebra of \mathcal{A}_k generated by \mathcal{S}_k . Since $\pi_k(\mathcal{S}_k) \subseteq \mathcal{S}_{k+1}$, we have that $\pi_k(C^*(\mathcal{S}_k)) \subseteq C^*(\mathcal{S}_{k+1})$.

Consider the following inductive system in \mathbf{C}^* :

$$C^*(\mathcal{S}_1) \xrightarrow{\pi_1} C^*(\mathcal{S}_2) \xrightarrow{\pi_2} C^*(\mathcal{S}_3) \xrightarrow{\pi_3} C^*(\mathcal{S}_4) \xrightarrow{\pi_4} \dots$$

Note that $\pi_{k,\infty}(C^*(\mathcal{S}_k)) \subseteq C^*(\mathcal{S}_\infty)$. We denote again by $\pi_{k,\infty}$ its restriction to $C^*(\mathcal{S}_k)$; note that it is a $*$ -homomorphism and $\pi_{k+1,\infty} \circ \pi_k = \pi_{k,\infty}$, $k \in \mathbb{N}$. We show that $C^*(\mathcal{S}_\infty) = C^*(\widehat{\mathcal{S}}_\infty)$, equipped with the family $\{\pi_{k,\infty}\}_{k \in \mathbb{N}}$, satisfies the universal property of the inductive limit $\varinjlim_{\mathbf{C}^*} C^*(\mathcal{S}_k)$ and therefore they are $*$ -isomorphic.

Suppose $(\mathcal{B}, \{\theta_k\}_{k \in \mathbb{N}})$ is a pair consisting of a C^* -algebra and a family of unital $*$ -homomorphisms $\theta_k : C^*(\mathcal{S}_k) \rightarrow \mathcal{B}$ such that $\theta_{k+1} \circ \pi_k = \theta_k$ for all $k \in \mathbb{N}$. Note that, if $s_1, \dots, s_n \in \mathcal{S}_\infty$ and $a = s_1 \cdots s_n$ then, writing $s_i = \pi_{k,\infty}(x_{k_i})$ for some $x_{k_i} \in \mathcal{S}_{k_i}$, $i = 1, \dots, n$, we have that $a = \pi_{k,\infty}(x_1 \cdots x_n)$. Suppose that

$$\pi_{k,\infty} \left(\sum_{i=1}^p x_1^i \cdots x_{n_i}^i \right) = \pi_{l,\infty} \left(\sum_{j=1}^q y_1^j \cdots y_{m_j}^j \right),$$

for some $k, l \in \mathbb{N}$, $x_s^i \in \mathcal{S}_k$ and $y_t^j \in \mathcal{S}_l$. Then

$$\lim_{d \rightarrow \infty} \left\| \pi_{k,d} \left(\sum_{i=1}^p x_1^i \cdots x_{n_i}^i \right) - \pi_{l,d} \left(\sum_{j=1}^q y_1^j \cdots y_{m_j}^j \right) \right\| = 0,$$

and letting $m = \max\{k, l\}$, we have that

$$\lim_{d \rightarrow \infty} \left\| \pi_{m,d} \left(\pi_{k,m} \left(\sum_{i=1}^p x_1^i \cdots x_{n_i}^i \right) - \pi_{l,m} \left(\sum_{j=1}^q y_1^j \cdots y_{m_j}^j \right) \right) \right\| = 0.$$

Thus,

$$\lim_{d \rightarrow \infty} \left\| \theta_d \circ \pi_{m,d} \left(\pi_{k,m} \left(\sum_{i=1}^p x_1^i \cdots x_{n_i}^i \right) - \pi_{l,m} \left(\sum_{j=1}^q y_1^j \cdots y_{m_j}^j \right) \right) \right\| = 0.$$

It follows that

$$\theta_k \left(\sum_{i=1}^p x_1^i \cdots x_{n_i}^i \right) = \theta_l \left(\sum_{j=1}^q y_1^j \cdots y_{m_j}^j \right).$$

Let

$$\mathcal{U} = \text{span} \left\{ \sum_{i=1}^p s_1^i \cdots s_{n_i}^i : p, n_i \in \mathbb{N}, s_m^i \in \mathcal{S}_\infty, k \in \mathbb{N} \right\}.$$

It follows from the previous paragraph that the map $\theta : \mathcal{U} \rightarrow \mathcal{B}$, given by

$$(46) \quad \theta \circ \pi_{k,\infty} = \theta_k, \quad k \in \mathbb{N},$$

is well-defined. It is clearly bounded, and we let $\tilde{\theta} : C^*(\mathcal{S}_\infty) \rightarrow \mathcal{B}$ be its continuous extension. Taking into account (46), we conclude that

$$(47) \quad C^*(\widehat{\mathcal{S}}_\infty) \cong \varinjlim_{C^*} C^*(\mathcal{S}_k).$$

By Theorem 6.5, $C_e^*(\mathcal{S}_k) = C^*(\mathcal{S}_k)$, and hence (the restriction of) π_k is a well-defined *-monomorphism from $C_e^*(\mathcal{S}_k)$ into $C_e^*(\mathcal{S}_{k+1})$; we can thus form the inductive system $(\{C_e^*(\mathcal{S}_k)\}_{k \in \mathbb{N}}, \{\pi_k\}_{k \in \mathbb{N}})$. Note that, by [24, Theorem 3.2],

$$(48) \quad C_e^*(\widehat{\mathcal{S}}_\infty) = \varinjlim_{C^*} C_e^*(\mathcal{S}_k);$$

we provide a direct argument for the equality (48) for the convenience of the reader. Namely, we show that $\varinjlim_{C^*} C_e^*(\mathcal{S}_k)$ satisfies the universal property of the C*-envelope $C_e^*(\mathcal{S}_\infty)$. Consider the following commuting diagram:

$$\begin{array}{ccccccccc} \mathcal{S}_1 & \xrightarrow{\phi_1} & \mathcal{S}_2 & \xrightarrow{\phi_2} & \mathcal{S}_3 & \xrightarrow{\phi_3} & \mathcal{S}_4 & \xrightarrow{\phi_4} & \dots \\ \iota_1 \downarrow & & \iota_2 \downarrow & & \iota_3 \downarrow & & \iota_4 \downarrow & & \\ C_e^*(\mathcal{S}_1) & \xrightarrow{\pi_1} & C_e^*(\mathcal{S}_2) & \xrightarrow{\pi_2} & C_e^*(\mathcal{S}_3) & \xrightarrow{\pi_3} & C_e^*(\mathcal{S}_4) & \xrightarrow{\pi_4} & \dots \end{array}$$

Note that we have denoted by ι_k the inclusion of \mathcal{S}_k into $C_e^*(\mathcal{S}_k)$. By Remark 4.15, there exists a unital completely order isomorphic embedding $\psi : \mathcal{S}_\infty \rightarrow \varinjlim_{\text{os}} C_e^*(\mathcal{S}_k)$ such that $\psi \circ \phi_{k,\infty} = \pi_{k,\infty} \circ \iota_k$, $k \in \mathbb{N}$. Observe that $\psi(\mathcal{S}_\infty)$ generates $\varinjlim_{C^*} C_e^*(\mathcal{S}_k)$; indeed, each $a_k \in C_e^*(\mathcal{S}_k)$ belongs to the span of elements of the form $s_1 \cdots s_n$, where $s_i \in \mathcal{S}_k$, $1 \leq i \leq n$. Thus, $\pi_{k,\infty}(a_k)$ belongs to the span of $\pi_{k,\infty}(s_1) \cdots \pi_{k,\infty}(s_n)$. It follows that $(\varinjlim_{C^*} C_e^*(\mathcal{S}_k), \psi)$ is a C*-cover of $\widehat{\mathcal{S}}_\infty$.

Suppose that (\mathcal{B}, α) is a C*-cover of \mathcal{S}_∞ . It follows that $\alpha \circ \pi_{k,\infty} : \mathcal{S}_k \rightarrow \mathcal{B}$ is a unital complete isometry for all $k \in \mathbb{N}$. Let \mathcal{B}_k be the C*-subalgebra of \mathcal{B} generated by $(\alpha \circ \pi_{k,\infty})(\mathcal{S}_k)$. Since $\alpha(\mathcal{S}_\infty)$ generates \mathcal{B} and $\bigcup_{k \in \mathbb{N}} \pi_{k,\infty}(\mathcal{S}_k)$ generates \mathcal{S}_∞ , we have that $\mathcal{B} = \bigcup_{k \in \mathbb{N}} \mathcal{B}_k$. By the universal property of the C*-envelope, for every $k \in \mathbb{N}$, there exists a unique *-homomorphism $\rho_k : \mathcal{B}_k \rightarrow C^*(\mathcal{S}_k)$ such that $\rho_k \circ \alpha \circ \phi_{k,\infty} = \iota_k$. Therefore

$$\pi_k \circ \rho_k \circ \alpha \circ \phi_{k,\infty} = \pi_k \circ \iota_k = \iota_{k+1} \circ \phi_k = \rho_{k+1} \circ \alpha \circ \phi_{k+1,\infty} \circ \phi_k = \rho_{k+1} \circ \alpha \circ \phi_{k,\infty},$$

for all $k \in \mathbb{N}$. Thus, $\pi_k \circ \rho_k = \rho_{k+1}$, $k \in \mathbb{N}$. We may thus construct the following commuting diagram:

$$\begin{array}{ccccccccc} \mathcal{B}_1 & \xrightarrow{\text{id}_{\mathcal{B}}} & \mathcal{B}_2 & \xrightarrow{\text{id}_{\mathcal{B}}} & \mathcal{B}_3 & \xrightarrow{\text{id}_{\mathcal{B}}} & \mathcal{B}_4 & \xrightarrow{\text{id}_{\mathcal{B}}} & \dots \\ \rho_1 \downarrow & & \rho_2 \downarrow & & \rho_3 \downarrow & & \rho_4 \downarrow & & \\ C_e^*(\mathcal{S}_1) & \xrightarrow{\pi_1} & C_e^*(\mathcal{S}_2) & \xrightarrow{\pi_2} & C_e^*(\mathcal{S}_3) & \xrightarrow{\pi_3} & C_e^*(\mathcal{S}_4) & \xrightarrow{\pi_4} & \dots \end{array}$$

By Theorem 2.14, there exists a *-homomorphism $\rho : \mathcal{B} \rightarrow \varinjlim_{C^*} C_e^*(\mathcal{S}_k)$ such that $\rho = \pi_{k,\infty} \circ \rho_k$ for all $k \in \mathbb{N}$. Note that

$$\rho \circ \alpha \circ \phi_{k,\infty} = \pi_{k,\infty} \circ \rho_k \circ \alpha \circ \phi_{k,\infty} = \pi_{k,\infty} \circ \iota_k = \psi \circ \phi_{k,\infty},$$

for all $k \in \mathbb{N}$. Therefore $\rho \circ \alpha = \psi$, and hence $\varinjlim_{\mathbf{C}^*} C_e^*(\mathcal{S}_k)$ is $*$ -isomorphic to the C^* -envelope of $\widehat{\mathcal{S}}_\infty$. It now follows from (47) and (48) that $C_e^*(\mathcal{S}_\infty) \cong C^*(\mathcal{S}_\infty)$. \square

6.2. Graphs associated to operator subsystems of UHF algebras.

The framework required to associate a graph with the UHF algebra $\widehat{\mathcal{A}}_\infty$ is established in [34]. We give some of its details here, since it will be needed in order to define graphs associated with operator subsystems of $\widehat{\mathcal{A}}_\infty$. Recall that $X_\infty = \Delta(\widehat{\mathcal{D}}_\infty)$ and $X_k = \Delta(\mathcal{D}_k)$, $k \in \mathbb{N}$. By Remark 6.2, $X_\infty = \varprojlim_{\mathbf{Top}} X_k$. For each $k \in \mathbb{N}$ and each $1 \leq i \leq n_k$, we have that $e_{i,i}^k \in \mathcal{D}_k \subseteq \widehat{\mathcal{D}}_\infty$. Let

$$X_i^k = \{x \in X_\infty : \langle x, e_{i,i}^k \rangle = 1\}.$$

Clearly, X_i^k is a closed and open subset of X_∞ such that, for all $k \in \mathbb{N}$,

$$X_\infty = \bigcup_{i=1}^{n_k} X_i^k.$$

We note that, if $[l_k]$ denotes the set $\{0, 1, 2, \dots, l_k - 1\}$, the space X_∞ is homeomorphic to the Cantor space $\Pi_{k=1}^\infty [l_k]$ (recall that $l_k = \frac{n_{k+1}}{n_k}$).

For each $k \in \mathbb{N}$ and each $1 \leq i, j \leq n_k$, let $\phi_{i,j}^k : C(X_i^k) \rightarrow C(X_j^k)$ be the $*$ -isomorphism given by $\phi_{i,j}^k(d) = e_{i,j}^{k*} d e_{i,j}^k$. Let $\alpha_{i,j}^k : X_j^k \rightarrow X_i^k$ be the homeomorphism induced by $\phi_{i,j}^k$; thus,

$$\langle \alpha_{i,j}^k(x), d \rangle = \langle x, \phi_{i,j}^k(d) \rangle, \quad x \in X_j^k, d \in C(X_i^k).$$

For $k \in \mathbb{N}$ and $1 \leq i, j \leq n_k$, let

$$E_{i,j}^k = \{(x, y) \in X_\infty \times X_\infty : x = \alpha_{i,j}^k(y) \text{ for some } y \in X_j^k\}$$

be the graph of the partial homeomorphism $\alpha_{i,j}^k$ of X_∞ . We have, equivalently,

$$E_{i,j}^k = \{(x, y) \in X_\infty \times X_\infty : \langle x, d \rangle = \langle y, e_{j,i}^k d e_{i,j}^k \rangle \text{ for all } d \in \mathcal{D}_k\}.$$

It will be convenient to write $R(e_{i,j}^k) = E_{i,j}^k$; for a subset \mathcal{E} of canonical matrix units in $\widehat{\mathcal{A}}_\infty$, we set $R(\mathcal{E}) = \cup_{e \in \mathcal{E}} R(e)$. In particular,

$$(49) \quad R(\widehat{\mathcal{A}}_\infty) = \bigcup \{E_{i,j}^k : k \in \mathbb{N}, 1 \leq i, j \leq n_k\}.$$

In Remark 6.9, whose statement is drawn from [34], we point out how the sets $E_{i,j}^k$ reflect the properties of the matrix units $e_{i,j}^k$. We set $E_{i,j}^{k*} = E_{j,i}^k$. For $E, F \subseteq X_\infty \times X_\infty$, let

$$(50) \quad E \circ F = \{(x, z) \in X_\infty \times X_\infty : \exists y \in X_\infty \text{ with } (x, y) \in E \text{ and } (y, z) \in F\}.$$

Remark 6.9. *The following hold, for any $k, m \in \mathbb{N}$ and any $1 \leq i \leq n_k$, $1 \leq j \leq n_m$:*

- (i) $E_{i,i}^k = \{(x, x) : x \in X_i^k\}$;
- (ii) $(x, y) \in E_{i,j}^k$ if and only if $(y, x) \in E_{j,i}^k$;
- (iii) $E_{i,j}^m \circ E_{j,k}^m = E_{i,k}^m$ and $E_{i,j}^m \circ E_{k,l}^m = \emptyset$ when $j \neq k$.

We have that $R(\widehat{\mathcal{A}}_\infty)$ is an equivalence relation on $X_\infty \times X_\infty$ and endows X_∞ with an associated graph. We define a topology on $R(\widehat{\mathcal{A}}_\infty)$ by specifying $\{E_{i,j}^k : k \in \mathbb{N}, 1 \leq i, j \leq n_k\}$ as a base of open sets. Note that each $E_{m,n}^l$ is either disjoint from $E_{i,j}^k$ or is a subset of $E_{i,j}^k$ (if the latter happens then $l > k$). Thus, this base consists of closed and open sets. Since X_∞ is compact, the sets $E_{i,j}^k$ are compact, too.

If $\widehat{\mathcal{S}}_\infty$ is an operator subsystem of $\widehat{\mathcal{A}}_\infty$, set

$$(51) \quad R(\widehat{\mathcal{S}}_\infty) = \bigcup \{E_{i,j}^k : e_{i,j}^k \in \widehat{\mathcal{S}}_\infty\}.$$

We specialise to the case of operator systems the Spectral Theorem for Bimodules from [34]. The following proposition follows from [34, Proposition 7.3 and Proposition 7.4].

Proposition 6.10. *Let $\widehat{\mathcal{S}}_\infty$ and $\widehat{\mathcal{T}}_\infty$ be concrete operator $\widehat{\mathcal{D}}_\infty$ -systems.*

- (i) *We have that $E_{i,j}^k \subseteq R(\widehat{\mathcal{S}}_\infty)$ if and only if $e_{i,j}^k \in \widehat{\mathcal{S}}_\infty$;*
- (ii) *If $R(\widehat{\mathcal{S}}_\infty) = R(\widehat{\mathcal{T}}_\infty)$ then $\widehat{\mathcal{S}}_\infty = \widehat{\mathcal{T}}_\infty$.*

Proposition 6.11. *Let $\widehat{\mathcal{S}}_\infty$ be a concrete operator $\widehat{\mathcal{D}}_\infty$ -system. Then $R(\widehat{\mathcal{S}}_\infty)$ is an open, reflexive and symmetric subset of $R(\widehat{\mathcal{A}}_\infty)$.*

Proof. We have that $R(\widehat{\mathcal{S}}_\infty)$ is open since it is a union of open sets. Since $\widehat{\mathcal{S}}_\infty$ contains the identity operator, $R(\widehat{\mathcal{S}}_\infty)$ is reflexive. Suppose that $(x, y) \in R(\widehat{\mathcal{S}}_\infty)$. Then there exists i, j, k such that $(x, y) \in E_{i,j}^k$ and $E_{i,j}^k \subseteq R(\widehat{\mathcal{S}}_\infty)$. By Proposition 6.10, $e_{i,j}^k \in \widehat{\mathcal{S}}_\infty$. Thus, $e_{j,i}^k = (e_{i,j}^k)^* \in \widehat{\mathcal{S}}_\infty$ and, again by Proposition 6.10, $E_{j,i}^k \subseteq R(\widehat{\mathcal{S}}_\infty)$. Thus, $(y, x) \in R(\widehat{\mathcal{S}}_\infty)$. \square

By Proposition 6.11, we may view $R(\widehat{\mathcal{S}}_\infty)$ is a (closed and) open subgraph of $R(\widehat{\mathcal{A}}_\infty)$. Conversely, if $P \subseteq R(\widehat{\mathcal{A}}_\infty)$ is an open, symmetric and reflexive subset, let

$$(52) \quad \mathcal{S}_\infty(P) = \overline{\text{span}}\{de_{i,j}^k f : d, f \in \widehat{\mathcal{D}}_\infty, E_{i,j}^k \subseteq P\}.$$

Theorem 6.12. *The map $P \rightarrow \mathcal{S}_\infty(P)$ is a bijective correspondence between the open subgraphs of $R(\widehat{\mathcal{A}}_\infty)$ and the concrete operator $\widehat{\mathcal{D}}_\infty$ -systems.*

Proof. The fact that, if P is an open subgraph of $R(\widehat{\mathcal{A}}_\infty)$ then $\mathcal{S}_\infty(P)$ is a concrete operator $\widehat{\mathcal{D}}_\infty$ -system follows easily from Remark 6.9. It remains to show that for any open reflexive and symmetric subset P of $R(\widehat{\mathcal{A}}_\infty)$, we have that $R(\mathcal{S}_\infty(P)) = P$. It is clear that $P \subseteq R(\mathcal{S}_\infty(P))$. Conversely, suppose that $E_{i,j}^k \subseteq R(\mathcal{S}_\infty(P))$, for some i, j and k with $i \neq j$. By Proposition 6.10,

$e_{i,j}^k \in \mathcal{S}_\infty(P)$. We claim that $E_{i,j}^k \subseteq P$; clearly, this claim will complete the proof.

Let $\tilde{\mathcal{A}}_p$ be the $\widehat{\mathcal{D}}_\infty$ -bimodule, generated by \mathcal{A}_p , $p \in \mathbb{N}$. By [34, Proposition 4.6], there exists a $\widehat{\mathcal{D}}_\infty$ -bimodule surjective projection $\Phi_p : \widehat{\mathcal{A}}_\infty \rightarrow \tilde{\mathcal{A}}_p$. Write

$$\mathcal{S}_0 = \text{span}\{de_{s,t}^p : d \in \widehat{\mathcal{D}}_\infty, E_{s,t}^p \subseteq P, p \in \mathbb{N}\}.$$

We have that $e_{i,j}^k = \lim_{m \rightarrow \infty} x_m$, for some $x_m \in \mathcal{S}_0$, $m \in \mathbb{N}$; thus,

$$e_{i,j}^k = \lim_{m \rightarrow \infty} \Phi_k(x_m).$$

Let $E_k = \cup_{u,v} E_{u,v}^k$. Then $\Phi_k(x_m) = \sum_{E_{s,t}^p \subseteq P \cap E^k} d_{s,t}^{p,m} e_{s,t}^p$, for some $d_{s,t}^{p,m} \in \widehat{\mathcal{D}}_\infty$ with $\text{supp}(d_{s,t}^{p,m}) \subseteq X_s^p$. It follows that

$$(53) \quad e_{i,j}^k = \lim_{m \rightarrow \infty} \sum_{E_{s,t}^p \subseteq P \cap E_{i,j}^k} d_{s,t}^{p,m} e_{i,j}^k.$$

Assume, by way of contradiction, that

$$Y \stackrel{\text{def}}{=} \cup\{E_{s,t}^p : E_{s,t}^p \subseteq P \cap E_{i,j}^k\} \neq E_{i,j}^k.$$

Letting $a \in \widehat{\mathcal{D}}_\infty$ be the projection corresponding to Y , we have that $a < e_{i,i}^k$ and $e_{i,j}^k = ae_{i,j}^k$, a contradiction. It follows that $Y = E_{i,j}^k$; since P is open, $E_{i,j}^k \subseteq P$. \square

Theorem 6.12 allows us to view the concrete operator $\widehat{\mathcal{D}}_\infty$ -systems as graph operator systems; we formalise this in the following definition.

Definition 6.13. Let $\widehat{\mathcal{A}}_\infty$ be a UHF algebra with canonical masa $\widehat{\mathcal{D}}_\infty$. An open, reflexive and symmetric subset of $R(\widehat{\mathcal{A}}_\infty)$ will be called a Cantor graph.

If P is a Cantor graph, the operator system $\mathcal{S}_\infty(P)$ defined in (52) will be called the Cantor graph operator system of P .

6.3. A graph isomorphism theorem. In this section, we prove a version of Theorem 6.6 for Cantor graph operator systems. Let $\widehat{\mathcal{A}}_\infty$ and $\widehat{\mathcal{B}}_\infty$ be UHF algebras with canonical masas $\widehat{\mathcal{D}}_\infty$ and $\widehat{\mathcal{E}}_\infty$, respectively, and let $X_\infty = \Delta(\widehat{\mathcal{D}}_\infty)$ and $Y_\infty = \Delta(\widehat{\mathcal{E}}_\infty)$. We write $e_{i,j}^k$ and $E_{i,j}^k$ (resp. $f_{i,j}^k$ and $F_{i,j}^k$) for the canonical matrix units of $\widehat{\mathcal{A}}_\infty$ (resp. $\widehat{\mathcal{B}}_\infty$) and their partial graphs.

Using the notation introduced in (50), for a set $P \subseteq R(\widehat{\mathcal{A}}_\infty)$, let

$$\epsilon(P) = \bigcup \{E_1 \circ \cdots \circ E_n : n \in \mathbb{N} \text{ and for each } j, E_j \subseteq P\}$$

$$\text{and } E_j = E_{s,t}^p \text{ for some } s, t, p\}.$$

Lemma 6.14. Let $\widehat{\mathcal{S}}_\infty$ be a concrete operator $\widehat{\mathcal{D}}_\infty$ -subsystem of $\widehat{\mathcal{A}}_\infty$. Then $\epsilon(R(\widehat{\mathcal{S}}_\infty)) = R(C^*(\widehat{\mathcal{S}}_\infty))$.

Proof. Write $P = R(\widehat{\mathcal{S}}_\infty)$ and $Q = \epsilon(P)$; it is clear that Q is the smallest open equivalence relation containing P . Note that $C^*(\mathcal{S}_\infty) = \mathcal{S}_\infty(Q)$; indeed, every canonical matrix unit in \mathcal{S}_∞ belongs to $\mathcal{S}_\infty(Q)$ and, since $\mathcal{S}_\infty(Q)$ is a C^* -algebra, $C^*(\mathcal{S}_\infty) \subseteq \mathcal{S}_\infty(Q)$. Suppose that $e_{i,j}^k \in \mathcal{S}_\infty(Q)$. By Theorem 6.12, $E_{i,j}^k \subseteq Q$; by compactness, $E_{i,j}^k$ is equal to a finite disjoint union of sets of the form $E_1 \circ \cdots \circ E_n$, where, for each j , the set E_j is a graph of a canonical partial homeomorphism contained in P . Thus, $e_{i,j}^k$ is equal to the sum of elements of the form $e_{i_1,j_1}^{k_1} \cdots e_{i_n,j_n}^{k_n}$, where $e_{i_r,j_r}^{k_r} \in \mathcal{S}_\infty \subseteq \widehat{\mathcal{S}}_\infty$. It follows that $\mathcal{S}_\infty(Q) \subseteq C^*(\mathcal{S}_\infty)$, and hence we have that $C^*(\mathcal{S}_\infty) = \mathcal{S}_\infty(Q)$. By Theorem 6.12, $Q = R(C^*(\widehat{\mathcal{S}}_\infty))$. \square

Theorem 6.15 below is an operator system version of a result of S. C. Power, [34, Theorem 7.5], characterising the isomorphism of limit algebras. Similarly to the case of operator subsystems of $\widehat{\mathcal{A}}_\infty$, one can define [34] a (closed and open) binary relation $R(A)$ (that is not necessarily a graph), associated to every subalgebra $A \subseteq \widehat{\mathcal{A}}_\infty$ with $\widehat{\mathcal{D}}_\infty \subseteq A$. Power shows in [34, Theorem 7.5] that if $\widehat{\mathcal{A}}_\infty$ and $\widehat{\mathcal{B}}_\infty$ are AF algebras with canonical masas $\widehat{\mathcal{D}}_\infty$ and $\widehat{\mathcal{E}}_\infty$, respectively, and if $A \subseteq \widehat{\mathcal{A}}_\infty$ and $B \subseteq \widehat{\mathcal{B}}_\infty$ are closed subalgebras, containing $\widehat{\mathcal{D}}_\infty$ and $\widehat{\mathcal{E}}_\infty$, respectively, then A and B are isometrically isomorphic via a map that sends $\widehat{\mathcal{D}}_\infty$ to $\widehat{\mathcal{E}}_\infty$ if and only if there exists a topological isomorphism between binary relations $R(A)$ and $R(B)$.

Theorem 6.15. *Let $\widehat{\mathcal{A}}_\infty$ and $\widehat{\mathcal{B}}_\infty$ be UHF-algebras with canonical masas $\widehat{\mathcal{D}}_\infty$ and $\widehat{\mathcal{E}}_\infty$, respectively. Set $X_\infty = \Delta(\widehat{\mathcal{D}}_\infty)$ and $Y_\infty = \Delta(\widehat{\mathcal{E}}_\infty)$. Let $P \subseteq X_\infty \times X_\infty$ and $Q \subseteq Y_\infty \times Y_\infty$ be Cantor graphs. The following are equivalent:*

- (i) *there exists a homeomorphism $\varphi : X_\infty \rightarrow Y_\infty$ such that $(\varphi \times \varphi)(P) = Q$;*
- (ii) *there exists a unital complete order isomorphism $\phi : \mathcal{S}_\infty(P) \rightarrow \mathcal{S}_\infty(Q)$ such that $\phi(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$.*

Proof. Set $\widehat{\mathcal{S}}_\infty = \mathcal{S}_\infty(P)$ (resp. $\widehat{\mathcal{T}}_\infty = \mathcal{S}_\infty(Q)$); then $\widehat{\mathcal{S}}_\infty$ is a concrete operator $\widehat{\mathcal{D}}_\infty$ -system (resp. a concrete operator $\widehat{\mathcal{E}}_\infty$ -system).

(i) \Rightarrow (ii) For ease of notation, set $\varphi^{(2)} = \varphi \times \varphi$. Let $\widetilde{P} = \epsilon(P)$ and $\widetilde{Q} = \epsilon(Q)$. As in the proof of [34, Theorem 7.5], $\varphi^{(2)}$ is a homeomorphism from \widetilde{P} onto \widetilde{Q} . By Lemma 6.14, $\widetilde{P} = R(C^*(\widehat{\mathcal{S}}_\infty))$ and $\widetilde{Q} = R(C^*(\widehat{\mathcal{T}}_\infty))$. Since $C^*(\widehat{\mathcal{S}}_\infty)$ (resp. $C^*(\widehat{\mathcal{T}}_\infty)$) is an AF C^* -algebra with a canonical masa $\widehat{\mathcal{D}}_\infty$ (resp. $\widehat{\mathcal{E}}_\infty$), by [34, Theorem 7.5], there exists a $*$ -isomorphism $\psi : C^*(\widehat{\mathcal{S}}_\infty) \rightarrow C^*(\widehat{\mathcal{T}}_\infty)$ such that $\psi(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$. We have that the restriction ϕ of ψ to $\widehat{\mathcal{S}}_\infty$ has its range in $\widehat{\mathcal{T}}_\infty$. By symmetry, ϕ is a bijection, and hence a unital complete order isomorphism.

(ii) \Rightarrow (i) By Remark 2.12, there exists a $*$ -isomorphism $\rho : C_e^*(\widehat{\mathcal{S}}_\infty) \rightarrow C_e^*(\widehat{\mathcal{T}}_\infty)$ which extends ϕ . By Theorem 6.8, $\rho : C^*(\widehat{\mathcal{S}}_\infty) \rightarrow C^*(\widehat{\mathcal{T}}_\infty)$ is a unital $*$ -isomorphism. Since $C^*(\widehat{\mathcal{S}}_\infty)$ and $C^*(\widehat{\mathcal{T}}_\infty)$ are subalgebras of $\widehat{\mathcal{A}}_\infty$ and $\widehat{\mathcal{B}}_\infty$, respectively, using [34, Theorem 7.5] we obtain a homeomorphism

$\varphi : X_\infty \rightarrow Y_\infty$ such that, if $\varphi^{(2)} = \varphi \times \varphi$ then the map

$$\varphi^{(2)} : R(C^*(\widehat{\mathcal{S}}_\infty)) \rightarrow R(C^*(\widehat{\mathcal{T}}_\infty))$$

is a homeomorphism and $R(\rho(e_{i,j}^k)) = \varphi^{(2)}(R(e_{i,j}^k))$ for any $e_{i,j}^k \in C^*(\widehat{\mathcal{S}}_\infty)$. Suppose that $E_{i,j}^k \subseteq P$. By Proposition 6.10, $e_{i,j}^k \in \widehat{\mathcal{S}}_\infty$. Since ϕ is a (complete) isometry, [34, Proposition 7.1], along with the compactness of Y_∞ shows that $\phi(e_{i,j}^k)$ is a sum of canonical matrix units. Moreover, by Theorem 6.12, $R(\phi(e_{i,j}^k)) \subseteq R(\widehat{\mathcal{T}}_\infty) = Q$. Thus, $\varphi^{(2)}(P) \subseteq Q$; by symmetry, $\varphi^{(2)}(P) = Q$. \square

We point out that the condition $\phi(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$ appearing in Theorem 6.15 (ii) is rather natural; indeed, since the algebra $\widehat{\mathcal{D}}_\infty$ uniquely determines X_∞ , this condition can be thought of as the requirement that the map ψ respect the “vertex sets” in the corresponding operator systems in order to give rise to a bona fide Cantor graph isomorphism.

6.4. A generalisation of Glimm’s theorem. We conclude this section with a generalised version of Glimm’s theorem (see [16]).

Theorem 6.16. *Let $\widehat{\mathcal{A}}_\infty$ and $\widehat{\mathcal{B}}_\infty$ be UHF algebras with canonical masas $\widehat{\mathcal{D}}_\infty$ and $\widehat{\mathcal{E}}_\infty$, respectively. Let $\widehat{\mathcal{S}}_\infty$ be a concrete operator $\widehat{\mathcal{D}}_\infty$ -system and $\widehat{\mathcal{T}}_\infty$ be a concrete operator $\widehat{\mathcal{E}}_\infty$ -system. The following are equivalent:*

- (i) *there exists a unital complete order isomorphism $\phi : \widehat{\mathcal{S}}_\infty \rightarrow \widehat{\mathcal{T}}_\infty$ such that $\phi(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$;*
- (ii) *there exist subsequences $(\mathcal{S}_{m_k})_{k \in \mathbb{N}}$ and $(\mathcal{T}_{n_k})_{k \in \mathbb{N}}$ of the sequences in the inductive systems associated with $\widehat{\mathcal{S}}_\infty$ and $\widehat{\mathcal{T}}_\infty$, respectively, and unital completely positive maps $\{\phi_k\}_{k \in \mathbb{N}}$ and $\{\psi_k\}_{k \in \mathbb{N}}$ such that*
 - (a) *the diagram*

$$\begin{array}{ccccccc} \mathcal{S}_1 & \longrightarrow & \mathcal{S}_{m_1} & \longrightarrow & \mathcal{S}_{m_2} & \longrightarrow & \cdots \\ \downarrow \phi_1 & \nearrow \psi_1 & \downarrow \phi_2 & \nearrow \psi_2 & \downarrow \phi_3 & \nearrow \psi_3 & \\ \mathcal{T}_{n_1} & \longrightarrow & \mathcal{T}_{n_2} & \longrightarrow & \mathcal{T}_{n_3} & \longrightarrow & \cdots \end{array}$$

commutes, and

- (b) $\phi_{k+1}(\mathcal{D}_{m_k}) \subseteq \mathcal{E}_{n_{k+1}}$ and $\psi_k(\mathcal{E}_{n_k}) \subseteq \mathcal{D}_{m_k}$, for all $k \in \mathbb{N}$.

Proof. (ii) \Rightarrow (i) By Remark 4.14, there exists a unital complete order isomorphism $\phi : \varinjlim \text{os } \mathcal{S}_k \rightarrow \varinjlim \text{os } \mathcal{T}_k$; let $\psi : \varinjlim \text{os } \mathcal{T}_k \rightarrow \varinjlim \text{os } \mathcal{S}_k$ be its inverse. Let $\widehat{\phi} : \widehat{\mathcal{S}}_\infty \rightarrow \widehat{\mathcal{T}}_\infty$ (resp. $\widehat{\psi} : \widehat{\mathcal{T}}_\infty \rightarrow \widehat{\mathcal{S}}_\infty$) be the (unital completely positive) extension of ϕ (resp. ψ). Clearly, $\widehat{\phi}$ and $\widehat{\psi}$ are each other’s inverses, and thus $\widehat{\mathcal{S}}_\infty$ and $\widehat{\mathcal{T}}_\infty$ are unital completely order isomorphic. Furthermore, condition (b) implies that $\widehat{\phi}(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$.

(i) \Rightarrow (ii) Suppose that $\phi : \widehat{\mathcal{S}}_\infty \rightarrow \widehat{\mathcal{T}}_\infty$ is a unital complete order isomorphism such that $\phi(\widehat{\mathcal{D}}_\infty) = \widehat{\mathcal{E}}_\infty$. By Remark 2.12, there exists a $*$ -isomorphism $\phi : C_e^*(\widehat{\mathcal{S}}_\infty) \rightarrow C_e^*(\widehat{\mathcal{T}}_\infty)$ extending ϕ . By Theorem 6.8, $\phi :$

$C^*(\widehat{\mathcal{S}}_\infty) \rightarrow C^*(\widehat{\mathcal{T}}_\infty)$ is a unital $*$ -isomorphism. By [34, Theorem 7.5], there exists a homeomorphism $\alpha : X_\infty \rightarrow Y_\infty$ such that $\alpha^{(2)} : R(C^*(\widehat{\mathcal{S}}_\infty)) \rightarrow R(C^*(\widehat{\mathcal{T}}_\infty))$ is a homeomorphism and $\alpha^{(2)}(E_{i,j}^k) = R(\phi(e_{i,j}^k))$. By Theorem 6.15 and its proof,

$$(54) \quad \alpha^{(2)}(R(\widehat{\mathcal{S}}_\infty)) = R(\widehat{\mathcal{T}}_\infty).$$

Set $\mathcal{L}_k = C^*(\mathcal{S}_k)$ and $\mathcal{M}_k = C^*(\mathcal{T}_k)$, $k \in \mathbb{N}$. By (48) and [34, Theorem 5.3] and its proof, there exist inductive systems of finite dimensional C^* -algebras and corresponding unital $*$ -homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{L}_1 & \longrightarrow & \mathcal{L}_{m_1} & \longrightarrow & \mathcal{L}_{m_2} & \longrightarrow & \cdots \\ \downarrow \phi_1 & \nearrow \psi_1 & \downarrow \phi_2 & \nearrow \psi_2 & \downarrow \phi_3 & \nearrow \psi_3 & \\ \mathcal{M}_{n_1} & \longrightarrow & \mathcal{M}_{n_2} & \longrightarrow & \mathcal{M}_{n_3} & \longrightarrow & \cdots \end{array}$$

The compactness of Y_∞ and [34, Proposition 7.1] show that the element $\phi_k(e_{i,j}^{m_k})$ is a sum of canonical matrix units. By passing to further subsequences if necessary, we may therefore assume that $\phi_k(\mathcal{S}_{m_k}) \subseteq \mathcal{T}_{n_{k+1}}$, $\psi_k(\mathcal{T}_{m_k}) \subseteq \mathcal{S}_{n_k}$, $\phi_k(\mathcal{D}_{m_k}) \subseteq \mathcal{E}_{n_{k+1}}$ and $\psi_k(\mathcal{E}_{n_k}) \subseteq \mathcal{D}_{m_k}$, for each k . Thus, conditions (a) and (b) are fulfilled. \square

Remark The concepts and questions studied in the present section have natural extended versions, whereas UHF algebras are replaced by more general AF algebras. We point out that the development of a general notion of AF operator systems will require substantial divergence from the context introduced in this paper. Indeed, here we have been only concerned with unital connecting maps, and we expect the passage to non-unital ones to lead to interesting technical considerations. We hope to pursue more general settings, than the UHF one studied in this section, in a future work.

We finish this section with an application of some of our results from Subsection 4.7. Let $G = (X, E)$ be a graph. A cycle of length n in G is an n -tuple (x_1, x_2, \dots, x_n) of distinct vertices such that $(x_i, x_{i+1}) \in E$ for each n (where addition is modulo n). A chord in the cycle (x_1, x_2, \dots, x_n) is an edge (x_i, x_j) with $|i - j| \geq 2$. The graph G is called chordal if every cycle in G of length at least four has a chord.

We recall that an operator system \mathcal{S} such that $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_c \mathcal{T}$ (up to a complete order isomorphism) for every operator system \mathcal{T} is called (min, c)-nuclear [20]. The operator system \mathcal{S} is called C^* -nuclear if $\mathcal{S} \otimes_{\min} A = \mathcal{S} \otimes_{\max} A$, for every C^* -algebra A .

Proposition 6.17. *Let $\mathcal{S}_{G_1} \xrightarrow{\phi_1} \mathcal{S}_{G_2} \xrightarrow{\phi_2} \mathcal{S}_{G_3} \xrightarrow{\phi_3} \mathcal{S}_{G_4} \xrightarrow{\phi_4} \cdots$ be an inductive system of graph operator systems. Suppose that G_k is chordal for each $k \in \mathbb{N}$. Then $\varinjlim \mathcal{S}_{G_k}$ is (min, c)-nuclear and C^* -nuclear.*

Proof. By [20, Proposition 6.11], $\mathcal{S}_{G_k} \otimes_{\min} \mathcal{T} = \mathcal{S}_{G_k} \otimes_c \mathcal{T}$, up to a complete order isomorphism. Since the minimal tensor product is injective, we have

that the map $\phi_k \otimes \text{id} : \mathcal{S}_{G_k} \otimes_c \mathcal{T} \rightarrow \mathcal{S}_{G_{k+1}} \otimes_c \mathcal{T}$ is a complete order embedding. By Theorem 4.38,

$$\left(\varinjlim_{\text{OS}} \mathcal{S}_{G_k} \right) \otimes_c \mathcal{T} = \varinjlim_{\text{OS}} (\mathcal{S}_{G_k} \otimes_c \mathcal{T}),$$

up to a complete order isomorphism. By Corollary 4.33,

$$\left(\varinjlim_{\text{OS}} \mathcal{S}_{G_k} \right) \otimes_{\min} \mathcal{T} = \varinjlim_{\text{OS}} (\mathcal{S}_{G_k} \otimes_{\min} \mathcal{T}),$$

up to a complete order isomorphism. The claim follows after another application of [20, Proposition 6.11].

By [20, Corollary 6.8], $\mathcal{S} \otimes_c A = \mathcal{S} \otimes_{\max} A$ for every operator system \mathcal{S} and every C^* -algebra A . It now follows from the previous paragraph that $\varinjlim_{\text{OS}} \mathcal{S}_{G_k}$ is C^* -nuclear. \square

Corollary 6.18. *Let P be a Cantor graph arising from a sequence $(G_k)_{k=1}^\infty$ of chordal graphs. Then $\mathcal{S}_\infty(P)$ is (\min, c) -nuclear and C^* -nuclear.*

Proof. Since $\mathcal{S}_\infty(P)$ is the completion of $\varinjlim_{\text{OS}} \mathcal{S}_{G_k}$, it suffices to show the following: if \mathcal{S} is a (\min, c) -nuclear operator system then its completion $\hat{\mathcal{S}}$ is also (\min, c) -nuclear. In order to see the latter statement, let \mathcal{T} be an operator system and suppose that $x \in (\hat{\mathcal{S}} \otimes_{\min} \mathcal{T})^+$. Then x belongs to the positive cone of the completion of $\mathcal{S} \otimes_{\min} \mathcal{T}$, and hence there exists a sequence $(x_i)_{i \in \mathbb{N}} \subseteq (\mathcal{S} \otimes_{\min} \mathcal{T})^+$ such that $x_i \rightarrow_{i \rightarrow \infty} x$. Since \mathcal{S} is (\min, c) -nuclear, $(x_i)_{i \in \mathbb{N}} \subseteq (\mathcal{S} \otimes_c \mathcal{T})^+$. Thus, $(x_i)_{i \in \mathbb{N}} \subseteq (\hat{\mathcal{S}} \otimes_c \mathcal{T})^+$ and hence $x \in (\hat{\mathcal{S}} \otimes_c \mathcal{T})^+$. A similar argument shows that

$$M_n(\hat{\mathcal{S}} \otimes_{\min} \mathcal{T})^+ = M_n(\hat{\mathcal{S}} \otimes_c \mathcal{T})^+, \quad n \in \mathbb{N},$$

and hence $\hat{\mathcal{S}} \otimes_{\min} \mathcal{T} = \hat{\mathcal{S}} \otimes_c \mathcal{T}$, up to a complete order isomorphism. \square

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